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ASPECTS OF BMS SYMMETRY

Doctoral Thesis

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Abstract

In this thesis we study different aspects of the asymptotic symmetries of the gravitational field as described by general relativity. We first briefly recall the surprising fact, uncovered by Bondi, van der Burg, Metzner and Sachs (BMS) [1]–[3] in the 1960s, that in the limit of a vanishing gravitational field the symmetry of spacetime does not reduce to the Poincaré group but is in fact described by an infinite-dimensional generalization thereof. In what follows this BMS group will be the central object of our interest.

The following chapter is devoted to the analysis of asymptotic symmetries in the Hamiltonian formulation of general relativity. We revisit previous treatments [4, 5] and find that at spatial infinity a symmetry which is even larger than BMS can be obtained.

It has been shown by Witten [6] in the 1980s that 2+1 dimensional gravity is completely equivalent to a specific gauge theory with Poincaré as gauge group. We investigate the question whether it is possible to obtain a gauge theory with BMS as gauge group.

Next we turn to the topic of the deformation of symmetries. After reviewing the mathematical notions of Hopf algebras and twist deformations we generalize the κ-deformation [7]–[11] of the Poincaré to the BMS algebra.

We then consider the topic of black hole entropy and revisit one of the earliest attempts of a microscopic explanation of its origin by 't Hooft [12]. By including backreaction effects we find a natural explanation for a certain regulator, which has been introduced ad-hoc by 't Hooft, and thereby remove the previous need to fine-tune its value.

Lastly, we review the information loss paradox and how it might be connected to the BMS symmetry [13] and point out that our results from the κ-deformation of the BMS algebra could be relevant in this context.

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Contents

1 Overview

Symmetries play a crucial role in the formulation of the fundamental theories of physics. A system is said to be symmetric if it is invariant under some transformation or a set thereof. If a system is, for instance, invariant under translations the outcome of any experiment is independent of where in space it is performed. Such a set of symmetry transformations is referred to as a symmetry group. In particle physics the underlying symmetry is the Poincaré group, which is the symmetry of spacetime in absence of a gravitational field and is fundamental for special relativity.

In this thesis we are concerned with the symmetries of spacetime in the presence of a gravitational field as described by general relativity (GR) and seen by an observer far away from the field's source. Such symmetries, which leave invariant the behavior of the spacetime metric only at large distances, are called asymptotic symmetries. They have been first studied by Bondi, van der Burg, Metzner and Sachs (BMS) [1]–[3] in the 1960s with the surprising result that asymptotically flat spacetimes exhibit close to null infinity, i.e. at large light-like distances, an asymptotic symmetry, the so-called BMS symmetry, which is much larger than the Poincaré group and is in fact an infinite-dimensional generalization thereof. Since BMS described the symmetry of spacetime in the limit of a vanishing gravitational field one can conclude that in the low-energy limit general relativity does not, in fact, reduce to just special relativity. The aim of this thesis is to develop a better understanding of this baffling, yet well-established, fact and to investigate its implications in a number of different contexts.

In section 2, which is based on parts of our publication [14], we explain how the BMS group emerges, in three and four-dimensional spacetimes, from analyzing the symmetries of asymptotically flat spacetimes and it is therefore the foundation which the subsequent sections will frequently refer to. Special attention will be paid to certain extensions of the original BMS group which have recently been proposed by Barnich and Troessaert [15]. We show that they contain an infinite number of Poincaré subgroups in four dimensions, a result that has previously been established in three dimensions only.

In section 3, which reproduces our preprint [16], we explore asymptotic symmetries in the Hamiltonian formulation of GR and in particular their relation to boundary terms present in the Hamiltonian. It was explained by Regge and Teitelboim [4] that such terms can be understood as Noether charges which generate the asymptotic symmetries of the system. Their analysis of asymptotically flat spaces lead to the result that the asymptotic symmetry of such spaces at spatial infinity, i.e. at large spacelike distances, is the Poincaré group and not an enlargement thereof. A natural question that arises is why the asymptotic symmetries at spatial and null infinity are so different. Only recently this question has been addressed, by Henneaux and Troessaert [5, 17], revealing that BMS might be the symmetry at spatial infinity after all. We are going to revisit their analysis and find that one can obtain an even larger symmetry group at spatial infinity.

Section 4 is about the relation of GR and gauge theories. The fundamental theories describing the electromagnetic, weak and strong interactions are formulated as gauge theories and at first sight they appear to have a structure which is entirely different from that of GR, at least in its standard formulation. However, in the so-called vielbein formulation many similarities appear and for the three-dimensional case Witten [6] has shown that GR is completely equivalent to a gauge theory based on the Poincaré group. We will investigate whether it is possible to generalize Witten's construction in order to obtain a gauge theory based on the BMS instead of the Poincaré group.

Section 5, which is based on our publications [14] and [58], is concerned with the deformation of symmetries. Such deformations are studied because they are thought [18] to shed some light on the elusive theory of quantum gravity [19], which seeks to describe gravity in regimes where quantum mechanical effects can not be neglected. Examples for such regimes are the center of a black hole and very early times in the standard model of cosmology, where in both cases GR predicts a divergent density. These divergences are commonly interpreted to signal a breakdown of GR and the need for a more fundamental theory of quantum gravity arises. Despite many efforts, to this day there exists no such theory which is broadly accepted and confirmed by experiment. Now, the κ-deformation [7]–[11] of the Poincaré algebra[1], a particular scheme of deformation, is believed to describe the symmetries of spacetime in the limit of a vanishing gravitational field at the Planck scale [20], which is the scale at which it is expected that quantum gravity effects can not be neglected. Since in the limit of a vanishing gravitational field the symmetry of spacetime is in fact much larger than the Poincaré group we are going to generalize the κ-deformation to the BMS group. As they are a prerequisite for performing this deformation we will first introduce a few mathematical notions, such as Hopf algebras and twist deformations.

Finally, in section 6 we first discuss Hawking radiation [21] and black hole entropy. The discussion of black hole entropy reproduces our publication [65]. When quantum effects are taken into account a black hole emits black body radiation at a temperature proportional to its inverse mass and it carries entropy proportional to the horizon area. What are the microscopical degrees of freedom responsible for this entropy? This is a long-standing question which still did not find a conclusive answer. Here we revisit one of the earliest attempts at an explanation by 't Hooft [12], who proposed that black hole entropy is the thermal entropy of a gas of quanta at Hawking temperature near the horizon. The novelty of our approach lies in the consideration of backreaction effects which have been neglected in the original treatment. The second part of this section is concerned with the information loss paradox [22] and a loophole [13] related to the BMS symmetry, recently proposed by Hawking, Perry and Strominger, which could potentially provide a solution. We are going to briefly explain what the paradox is about and describe an ongoing discussion in the literature about the validity of aforementioned loophole. Lastly, we show that in fact our results from the κ-deformation of BMS enter this discussion and explain the implications.

[1] Our discussion will mainly be on the level of the Lie algebra corresponding to the symmetry group, which can be thought of as describing infinitesimal symmetry transformations. The Lie algebra is the tangent space at the identity of the Lie group.

2 Symmetries of asymptotically flat spacetimes

In this section, which partially reproduces our publication [14], we will present the BMS algebra and discuss its structure. We start with the four-dimensional case, which is more familiar, and in the next subsection we turn to the simpler three-dimensional case.

2.1 Four-dimensional gravity

2.1.1 Bondi coordinates, boundary conditions and BMS algebra

In an effort to better understand gravitational waves in the full, non-linear theory of GR, BMS [1] [2] [3] investigated solutions to Einstein equations which consist of an isolated source and which become flat at null infinity, see figure 1. The definition of asymptotic flatness employed by BMS is that in the limit of large r, while keeping all other coordinates constant, the spacetime metric behaves as

$$\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(r^{-1}), \tag{2.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. We are going to use the same definition throughout and only mention that there exist other, coordinate independent, definitions, see [23], [24].

Naively one could expect that the symmetry group, as seen by an observer close to null infinity, of such asymptotically flat spacetimes is just the Poincaré group. The surprising result of BMS was that the symmetry transformations that leave the form of an asymptotically flat metric invariant do not just include translations, rotations and boosts but also so-called supertranslations, which can be thought of as angle-dependent translations. These asymptotic symmetry transformations form the BMS group, which is a semi-direct product of the Lorentz group with an infinite-dimensional abelian group of supertranslations.

The analysis of BMS has been carried out using coordinates (u, r, x^A) in the Bondi gauge

$$g^{uu} = 0, \quad g^{uA} = 0, \quad \det g_{AB} = r^4 \det \gamma_{AB}, \tag{2.2}$$

where $u = t - r$ is the retarded time, x^A are angular coordinates and $\det \gamma_{AB}$ is the determinant of the unit sphere metric. The first condition implies that the normal vector of hypersurfaces defined by $u = const.$, $n^\mu = g^{\mu\nu}\partial_\nu u$, is null and thus u is labeling null hypersurfaces. Angular coordinates x^A are defined such that the directional derivative along n^μ vanishes, $n^\mu \partial_\mu x^A = 0$, and r is defined such that the area of a 2-surface $u = const., r = const$ is $4\pi r^2$. After lowering the indices these conditions are $g_{rr} = g_{rA} = 0$.

Using these Bondi coordinates, BMS write the spacetime metric in the form [2]

$$ds^2 = -\,Ue^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 g_{AB}\left(dx^A - \mathcal{U}^A du\right)\left(dx^B - \mathcal{U}^B du\right), \tag{2.3}$$

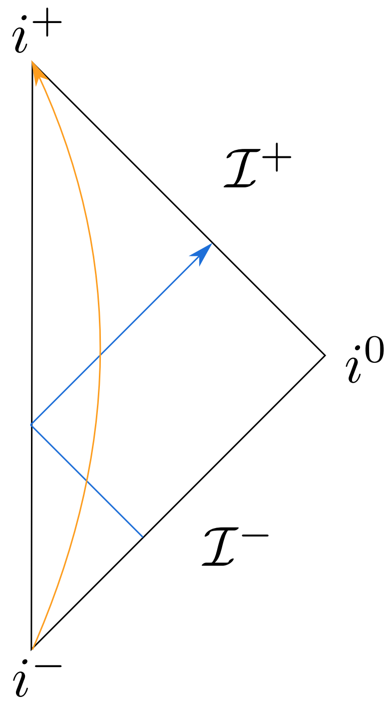

Figure 1: Penrose diagram of Minkowski spacetime. A null geodesic is presented in blue and a timelike geodesic in orange. We denote with i^+, i^- future and past timelike infinity, $\mathcal{I}^+, \mathcal{I}^-$ future and past null infinity and i^0 spatial infinity. Intuitively one can think of $\mathcal{I}^-$ as the region where null geodesics originate and of $\mathcal{I}^+$ as the region where they end.

where U, β and $\mathcal{U}^A$ are functions of u, r, x^A.

In line with their analysis, asymptotically flat spacetimes at null infinity are defined by assuming the following expansion of metric functions, see also [15, 25, 26]

$$\begin{aligned} \beta &= \frac{\beta_0}{r} + \frac{\beta_1}{r^2} + O\left(r^{-3}\right) \\ U &= 1 - \frac{2m_B}{r} + O\left(r^{-2}\right) \\ \mathcal{U}^A &= \frac{1}{r^2} U^A + \frac{1}{r^3}\left[-\frac{2}{3}N^A + \frac{1}{16}D^A\left(C_{BC}C^{BC}\right)\right. \\ &\left. + \frac{1}{2}C^{AB}D^C C_{BC}\right] + O\left(r^{-4}\right) \\ g_{AB} &= \gamma_{AB} + \frac{1}{r}C_{AB} + \frac{1}{r^2}D_{AB} + O(r^{-3}), \end{aligned} \tag{2.4}$$

where γ_{AB} is the round metric on the unit 2-sphere and D^A the covariant derivative associated with γ_{AB}. We thereby define boundary conditions which describe the large-distance behavior of the metric. The function $m_B(u, x^A)$ is referred to as Bondi mass aspect, $N_A(u, x^A)$ as the angular momentum aspect and the retarded time-derivative $N_{AB}(u, x^A) = \partial_u C_{AB}(u, x^A)$ as Bondi news. In the literature exist several different definitions of N_A, see [15, 25, 26, 27], where for definiteness we cite the one given in [26].

Imposing the above determinant condition, i.e. the last condition in (2.2), makes C_{AB}

traceless. To see this we first act with the r-derivative

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0, \tag{2.5}$$

and then consider

$$\begin{aligned}\det\left(\frac{g_{AB}}{r^2}\right) &= \det\left(\gamma_{AB} + \frac{C_{AB}}{r} + \mathcal{O}(r^{-2})\right)\\ &= \det\gamma\left(1 + \frac{\gamma^{AB}C_{AB}}{r} + \mathcal{O}(r^{-2})\right).\end{aligned} \tag{2.6}$$

Now, acting with the r-derivative on the above expression yields

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = -\det\gamma\frac{\gamma^{AB}C_{AB}}{r^2} + \mathcal{O}(r^{-3}) \tag{2.7}$$

and vanishing of the leading order implies that C_{AB} is traceless. Since C_{AB} is traceless and symmetric it has two polarization modes and contains the information about gravitational radiation near $\mathcal{I}^+$.

Imposing Einstein field equations in vacuum[2], $R_{\mu\nu} = 0$, one obtains the following conditions

$$U_A = -D^B C_{AB}/2, \quad \beta_0 = 0, \quad \beta_1 = -\frac{1}{32}C_{AB}C^{AB} \tag{2.8}$$

and the metric now reads

$$\begin{aligned}ds^2 =& -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B\\ &+ \frac{2m_B}{r}du^2 + rC_{AB}dx^A dx^B + D^B C_{AB}dudx^A + \frac{1}{16r^2}C_{AB}C^{AB}dudr\\ &+ \frac{1}{r^3}\left[\frac{4}{3}N_A - \frac{1}{8}D_A\left(C_{BC}C^{BC}\right) - C^B_A D_C C_{BC}\right]dudx^A\\ &+ (\text{subleading terms}).\end{aligned} \tag{2.9}$$

Since the boundary conditions (2.4) allow the g_{uA} component to be of order $\mathcal{O}(r^0)$ it might seem that the metric is not asymptotically flat, since in the large r limit this component is not suppressed. Transforming to Cartesian coordinates reveals that this component is in fact subleading and instead of explicitly performing this transformation one can argue directly from the form of the metric (2.9) that this has to be the case. Since the first line just gives the diagonal matrix $\text{diag}(-1,1,1,1)$ when transforming to Cartesian coordinates y^μ we can infer that $dr \sim du \sim \mathcal{O}(r^0)\cdot dy^\mu$, $dx^A \sim \mathcal{O}(r^{-1})\cdot dy^\mu$ and it is therefore clear that the asymptotic flatness of (2.9) is not spoiled by the g_{uA} component.

[2] For simplicity we only consider pure gravity. Allowing a non-vanishing energy-momentum does not change our discussion of asymptotic symmetries, see [26] for details.

Before proceeding with the discussion of the asymptotic symmetries, we note that one of the central results of [1] is that the mass of the system, $M(u) = \int d^2\Omega m_B(u, x^A)$, is constant only for vanishing Bondi news and it is otherwise a monotonically decreasing function of u. This has the physical interpretation that the Bondi news describes the rate of gravitational radiation which, when non-vanishing, carries away mass through null infinity.

We are now describing how the asymptotic symmetries of the metric (2.9) are obtained. Since the Lie derivative tells us how the metric changes under a transformation of coordinates $x^\mu \to x^\mu + \xi^\mu$ we can find transformations which preserve the Bondi gauge by the following demands on the Lie derivative

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad \mathcal{L}_\xi \partial_r \det\left(g_{AB}/r^2\right) = 0. \tag{2.10}$$

Similarly we find transformations which leave the asymptotic form of the metric (2.9) invariant by demanding

$$\mathcal{L}_\xi g_{uu} = \mathcal{O}\left(r^{-1}\right), \quad \mathcal{L}_\xi g_{ur} = \mathcal{O}\left(r^{-2}\right), \quad \mathcal{L}_\xi g_{uA} = \mathcal{O}\left(r^{0}\right), \quad \mathcal{L}_\xi g_{AB} = \mathcal{O}(r). \tag{2.11}$$

Solving these requirements one obtains the BMS generators in terms of two functions on the sphere $f(z, \bar{z})$ and $R^A(z, \bar{z})$, see also [15, 28]

$$\begin{aligned} \xi(f, R) = &\left[f + \frac{u}{2} D_A R^A + o\left(r^0\right)\right] \partial_u \\ &+ \left[R^A - \frac{1}{r} D^A f + o\left(r^{-1}\right)\right] \partial_A \\ &+ \left[-\frac{r+u}{2} D_A R^A + \frac{1}{2} D_A D^A f + o\left(r^0\right)\right] \partial_r \end{aligned} \tag{2.12}$$

which can be understood as "asymptotic Killing vectors" since they do not leave the metric invariant everywhere but only at the boundary. While f is unconstrained the last condition in (3.29) imposes that R^A has to obey the conformal Killing equation on the 2-sphere

$$D_A R_B + D_B R_A = \gamma_{AB} D_C R^C. \tag{2.13}$$

For our further discussion it is helpful to introduce a specific choice of angular coordinates. We will use complex stereographic coordinates $x^A = (z, \bar{z})$, which are related to the standard coordinates (θ, ϕ) via $z = e^{i\phi} \cot(\theta/2)$, $\bar{z} = z^*$. The unit round metric γ_{AB} reads in these coordinates $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$.

In these coordinates the equation (2.13) imposes the conditions $\partial_z R^{\bar{z}} = 0$ and $\partial_{\bar{z}} R^z = 0$, so that $R^z(z)$ is an holomorphic function and $R^{\bar{z}}(\bar{z})$ antiholomorphic. If we demand that $\xi(f, R)$ are to be globally well-defined there are further restrictions on R^A. To see this we expand R^z in monomials z^{n+1} and consider the vector $v_n = z^{n+1}\partial_z$ which is the z-component of ξ. The vector v_n is divergent at $z = 0$ for $n < -1$ and it is divergent in the limit $z \to \infty$ for $n > 1$, where $z = 0$ corresponds to the north pole and $z \to \infty$ to the south pole on the 2-sphere. It is clear that in both limits v_n is finite at $n = -1$ but to demonstrate that it is also finite for $n = 0, 1$ we take the limit $z \to \infty$ by introducing $w = z^{-1}$ and setting $w = 0$ and find $v_n = -\omega^{-n+1}\partial_\omega$, which is indeed finite at $\omega = 0$ for $n = 0, 1$. By the same argument one finds three more globally well-defined functions for $R^{\bar{z}}$ and below we show that the Killing vectors $\xi(0, R)$ with $R^z = z^{n+1}, R^{\bar{z}} = \bar{z}^{n+1}$, $n = \pm 1, 0$ are in fact the generators of Lorentz transformations.

In leading order these Killing vectors form the four-dimensional BMS algebra $\mathcal{B}^4$, which is an infinite-dimensional algebra with the bracket [28]

$$\xi(\hat{f}, \hat{R}) = \left[\xi(f, R), \xi(f', R')\right], \tag{2.14}$$

where

$$\begin{aligned} \hat{f} &= R^A D_A f' + \frac{1}{2} f D_A R'^A - R'^A D_A f - \frac{1}{2} f' D_A R^A \\ \hat{R}^A &= R^B D_B R'^A - R^B D_B R^A. \end{aligned} \tag{2.15}$$

It can be seen from (2.14) that $T = \xi(f, 0)$ form an abelian ideal of the algebra. These generators are referred to as supertranslations since they generalize the Poincaré translations. The generators $l = \xi(0, R)$ contain the Lorentz algebra and possible extensions of it which will be referred to as superrotations.

It is interesting to observe what happens when one acts with a supertranslation on the Minkowski vacuum, which is characterised by $m_B = C_{zz} = N_{zz} = 0$, and in which case one finds [27]

$$\mathcal{L}_f m_B = 0, \quad \mathcal{L}_f N_{zz} = 0, \quad \mathcal{L}_f C_{zz} = -2D_z^2 f. \tag{2.16}$$

The supertranslated vacuum still has zero Bondi mass and Bondi News but it obtains a finite C_{zz} term. Furthermore, for the curvature to vanish we need to have [27]

$$C_{zz} = -2D_z^2 C(z, \bar{z}), \tag{2.17}$$

where the function C transforms as

$$\mathcal{L}_f C = f. \tag{2.18}$$

The fact that the supertranslations, despite being an asymptotic symmetry, do not leave the vacuum invariant can be viewed as the spontaneous breaking of that symmetry and C is labeling inequivalent gravitational vacua, see also [29]. We will return to this important point later in this section where we observe an analogous effect on the level of Poincaré subalgebras of $\mathcal{B}^4$.

For our further discussion it will be useful to expand supertranslations and superrotations in the basis of z, $\bar{z}$ monomials

$$f_{mn} = \frac{z^m \bar{z}^n}{1 + z\bar{z}}, \quad R_n^z = -\, z^{n+1}, \quad R_n^{\bar{z}} = -\bar{z}^{n+1}. \tag{2.19}$$

In terms of the basis vectors $T_{mn} = \xi(f_{mn}, 0)$, $l_n = \xi(0, R_n^z)$, $\bar{l}_n = \xi(0, R_n^{\bar{z}})$ the algebra (2.14) takes the form

$$\begin{aligned} [l_m, l_n] &= (m-n) l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0, \\ [l_l, T_{m,n}] &= \left(\frac{l+1}{2} - m\right) T_{m+l,n}, \quad [\bar{l}_l, T_{m,n}] = \left(\frac{l+1}{2} - n\right) T_{m,n+l}. \end{aligned} \tag{2.20}$$

There exist different versions of this algebra in the literature, as is explained in detail in the next subsection, and which values for l, m, n are allowed depends on which version one chooses.

One can identify the Poincaré translational generators in Cartesian coordinates as

$$\begin{aligned} P_0 + P_3 = T_{11}, \quad P_0 - P_3 = T_{00}, \\ P_1 = T_{10} + T_{01}, \quad P_2 = i(T_{10} - T_{01}), \end{aligned} \tag{2.21}$$

and the boosts and rotations as

$$\begin{aligned} J_1 &= -\frac{1}{2}(l_1 + l_{-1} + \bar{l}_1 + \bar{l}_{-1}), \quad K_1 = \frac{i}{2}(l_1 + l_{-1} - \bar{l}_1 - \bar{l}_{-1}) \\ J_2 &= \frac{i}{2}(l_1 - l_{-1} + \bar{l}_1 + \bar{l}_{-1}), \quad K_2 = \frac{1}{2}(l_1 - l_{-1} + \bar{l}_1 - \bar{l}_{-1}) \\ J_3 &= l_0 + \bar{l}_0, \quad K_3 = l_0 - \bar{l}_0 \end{aligned} \tag{2.22}$$

and the elements $\{l_0, l_{\pm 1}, \bar{l}_0, \bar{l}_{\pm 1} T_{00}, T_{11}, T_{10}, T_{01}\}$ therefore form a subalgebra isomorphic to the Poincaré algebra.

2.1.2 Extensions of BMS and Poincaré subalgebras

Historically, the BMS algebra $\mathcal{B}^4_{\text{BMS}}$ of Bondi, van der Burg, Metzner, and Sachs was first defined as the semi-direct sum of the infinite-dimensional abelian algebra of supertranslations $\mathbf{s}$ and the Lorentz algebra $\mathbf{so}(3,1)$, i.e. $l_n, \bar{l}_n$ with $n = 0, \pm 1$ and $T_{p,q}$ with $p, q \in \mathbb{N}$

$$\mathcal{B}^4_{\text{BMS}} = \mathbf{so}(3,1) \oplus_S \mathbf{s}. \tag{2.23}$$

This corresponds to the above mentioned case of globally well-defined Killing vectors.

Two extensions of the $\mathcal{B}^4_{\text{BMS}}$ algebra have been proposed in recent years. In [15] the algebra is extended by including all l_n, $n \in \mathbb{Z}$, which implies the inclusion of $m, n \in \mathbb{Z}$ the labels of $T_{m,n}$ as well. The generators of the extended Lorentz algebra l_n are referred to as superrotations. The resulting algebra is referred to as extended BMS algebra and is the semi-direct sum of two copies of infinitesimal diffeomorphisms on S^1 and extended supertranslations $\mathbf{s}^*$

$$\mathcal{B}^4_{\text{ext}} = \Big(\text{Diff}(S^1) \oplus \text{Diff}(S^1)\Big) \oplus_S \mathbf{s}^*, \tag{2.24}$$

with commutation relations still given by (2.20).

In [30] a different extension has been proposed in which R^A is not constrained by the conformal Killing equation. To this end the condition (3.29) is softened to

$$\mathcal{L}_\xi g_{AB} = O(r^2) \tag{2.25}$$

which allows the leading order of g_{AB} to fluctuate such that it is not identical to the round metric anymore. In particular this means that in the large r limit one does not obtain the Minkowski metric and one is thus working with a different notion of asymptotic flatness, which is explained in detail in [30]. Since R^A is not constrained the resulting algebra is

$$\mathcal{B}^4_{\text{gen}} = \text{Diff}(S^2) \oplus_S \mathbf{s} \tag{2.26}$$

and is referred to as generalized BMS algebra.

Contrary to the classical BMS algebra $\mathcal{B}^4_{\text{BMS}}$ the extended BMS algebra (2.20) contains an infinite number of (overlapping) subalgebras generated by the sets of 10 generators

$$\Big\{ l_0, l_{\pm(1-2n)}, \bar{l}_0, \bar{l}_{\pm(1-2n)}, T_{n,n}, T_{1-n,n}, T_{n,1-n}, T_{1-n,1-n} \Big\}$$

with $n \in \mathbb{Z}$, which are isomorphic to the Poincaré algebra after rescaling

$$l_i \to \frac{l_i}{m}, \quad \bar{l}_i \to \frac{\bar{l}_i}{m}, \tag{2.27}$$

where $m = 1 - 2n$. Note, however, that this is not an automorphism on the entire $\mathcal{B}^4_{\text{ext}}$, e.g.

$$[l'_p, l'_q] = \frac{1}{m}(p-q) l'_{p+q}, \quad l'_p = \frac{1}{m} l_p, \tag{2.28}$$

which coincides with (2.20) only for $m = 1$. These Poincaré subalgebras will play an important role in section 5.4 which is concerned with the κ-deformation of $\mathcal{B}^4_{\text{ext}}$. As we are going to show, these subalgebras lead to a family of so-called twist deformations parametrized by n.

It is tempting to identify these Poincaré subalgebras with the aforementioned inequivalent vacua. But notice that these subalgebras only exist when one allows for superrotations whereas the inequivalent vacua are obtained from the original vacuum with $C = 0$ purely by supertranslations so this identification is not completely clear. As we are going to see in the next section these subalgebras also exist in the 3 dimensional case and we are able to show explicitly that they are the isometries of inequivalent vacuum solutions.

2.1.3 BMS charges

Before we continue with the discussion of the three-dimensional case we mention, for later reference, some facts about charges associated with the BMS symmetry. It is well known that every continuous symmetry of a theory is associated with conserved charges, see [31] for an interesting review of Noether's theorem. In a Hamiltonian formulation these charges generate the symmetry transformation via the Poisson brackets. The same holds true for the asymptotic symmetries in general relativity, see e.g. [4], with the peculiarity that the associated charges are given by surface integrals (instead of volume integrals as is generically the case in a field theory). This has to do with the fact that as a consequence of the general diffeomorphism invariance of GR the bulk part of the Hamiltonian vanishes as a constraint. We will have a lot more to say about the Hamiltonian formulation of GR in section 3.

Since we are interested in asymptotic symmetries at null infinity the charges should be given as surface integrals at null infinity. This is problematic since due to the presence of radiative fluxes the charges are in general not actually conserved, which has spawned a number of proposals to resolve this problem, e.g. [25, 32, 33]. Owing to the different approaches to this problem in the literature one can find slightly varying expressions for the BMS charges but for concreteness we cite here [26] (this reference also discusses the other expressions). Supertranslation and superrotation charges are given therein as

$$\begin{aligned} Q =& \frac{1}{16\pi} \int d^2\Omega \Big[4 f m_B - 2u_0 R^A D_A m_B + 2 R^A N_A \\ & - \frac{1}{8} R^A D_A \left(C_{BC} C^{BC} \right) - \frac{1}{2} R^A C_{AB} D_C C^{BC} \Big] . \end{aligned} \tag{2.29}$$

2.2 Three-dimensional gravity

2.2.1 Fall-off conditions and BMS algebra

In the case of three-dimensional gravity the discussion of asymptotically flat spacetimes is similar to the four-dimensional case. One should be aware, however, that despite the formal similarities of three-dimensional and four-dimensional gravity they are in fact very different theories. The main difference is that in three-dimensional gravity with a vanishing cosmological constant every vacuum solution is flat, which is a consequence of the fact that the Weyl tensor, which describes the only part of curvature that can exist in vacuum, vanishes identically, see e.g. [34]. In particular this implies that gravitational radiation does not exist in three-dimensional gravity.

Nonetheless one can impose boundary conditions which define asymptotically flat spacetimes and investigate the asymptotic symmetries, just as we did in the four-dimensional case. The boundary conditions are chosen to be [28]

$$\begin{aligned} ds^2 = &\left(-1+\frac{\bar{g}_{uu}}{r}\right)du^2 + 2\left(-1+\frac{\bar{g}_{ur}}{r}\right)dudr \\ &+\left(h(u,z)+\frac{\bar{g}_{uz}}{r}\right)dudz - \left(\frac{r^2}{z^2}+\bar{g}_{zz}r\right)dz^2 + \text{subleading}, \end{aligned} \tag{2.30}$$

where the barred metric components are functions of $u = t-r$ and $z = e^{i\phi}$. The asymptotic Killing vectors for supertranslations can be parametrized by z^n and read [35]

$$\begin{aligned} T_n =& z^n\partial_u - nz^{n+1}\int_r^\infty \frac{dr'}{r'^2}g_{r'u}\partial_z \\ &+\left(rnz^n\int_r^\infty \frac{dr'}{r'^2}(ng_{r'u}+\partial_z g_{r'u}) + \frac{nz^{n+1}}{r}g_{zu}\right)\partial_r, \end{aligned} \tag{2.31}$$

where the unbarred metric components are the full, r-dependent components, and using their large r behavior according to (2.30) the supertranslation generators can be written as

$$T_n = z^n\partial_u + \left(\frac{nz^{n+1}}{r} + O(1/r^2)\right)\partial_z - \left(n^2z^n + O(1/r)\right)\partial_r. \tag{2.32}$$

For the superrotations the Killing vectors are, using the same parametrization,

$$\begin{aligned} l_n =& i\Bigg(nuz^n\partial_u - \left(nrz^n - uzn^2\partial_z\left[z^n\int\frac{dr'}{r'^2}g_{r'u}\right] + \frac{n^2z^{n+1}}{r}g_{uz}\right)\partial_r \\ &+ z^{n+1}\left(1-un^2\int\frac{dr'}{r'^2}g_{r'u}\right)\partial_z\Bigg) \end{aligned} \tag{2.33}$$

and using the expansion at large r

$$l_n = iz^n\left(nu\partial_u - \left(rn + O(r^0)\right)\partial_r + \left(z+O(r^{-1})\right)\partial_z\right). \tag{2.34}$$

Setting $n = 0, \pm 1$ one obtains the generators of the three standard Poincaré translations, one rotation and two boosts which are related as

$$\partial_t = T_0, \quad \partial_x = T_1 + T_{-1}, \quad \partial_y = i(T_1 - T_{-1}) \tag{2.35}$$

$$\partial_\phi = l_0, \quad x\partial_t - t\partial_x = l_1 + l_{-1}, \quad y\partial_t - t\partial_y = i(l_1 - l_{-1}). \tag{2.36}$$

At leading order the Killing vectors, under the standard Lie bracket, form a representation of the abstract infinite-dimensional BMS3 algebra with $m, n \in \mathbb{Z}$

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [l_m, T_n] = (m-n)T_{m+n}\,, \quad [T_m, T_n] = 0, \tag{2.37}$$

which we refer to as $\mathcal{B}^3$.

Notice, that in order for the Killing vectors to form a faithful representation of the BMS algebra at all orders the Lie bracket has to be modified [28]. This modification arises because the T_m and l_m leave the metric only asymptotically invariant and since they themselves depend on the metric components one has to take into account the change of the Killing vectors induced by the change in the metric.
We now introduce, for convenience, lightcone coordinates which are related to Cartesian coordinates by

$$x^+ = \frac{x^0 + x^2}{\sqrt{2}}\,, \quad x^- = \frac{x^0 - x^2}{\sqrt{2}}\,, \quad x^1 = x^1 \tag{2.38}$$

and the three-dimensional Poincaré algebra in lightcone coordinates reads

$$[M_{+1}, M_{-1}] = i\eta_{11} M_{+-}\,, \quad [M_{+-}, M_{\pm 1}] = \mp i\eta_{+-} M_{\pm 1}\,, \tag{2.39}$$

$$[M_{+-}, P_\pm] = \mp i\eta_{+-} P_\pm\,, \quad [M_{\pm 1}, P_1] = -i\eta_{11} P_\pm\,, \tag{2.40}$$

$$[M_{\pm 1}, P_\mp] = i\eta_{+-} P_1\,, \tag{2.41}$$

where $\eta_{+-} = \eta_{-+}$ and η_{11} denote Lorentzian metric components, cf. [36].

2.2.2 Poincaré subalgebras and corresponding vacua

We find that for each $n = 1, 2, \cdots \in \mathbb{N}$ there is an embedding of (2.39)-(2.41) to the infinite-dimensional Lie algebra (2.37)

$$M_{+-} = -\imath l_0\,, \quad M_{\pm 1} = \pm \frac{\imath}{\sqrt{2}} l_{\pm n}\,, \tag{2.42}$$

$$P_1 = -\imath T_0, \quad P_\pm = \frac{\imath}{\sqrt{2}} T_{\pm n}\,, \tag{2.43}$$

if one identifies $\eta_{+-} = \eta_{-+} = -\eta_{11} = n$. Consider for instance the following commutator and substitute according to the above embedding

$$[M_{+-}, M_{-1}] \to \frac{1}{2}[l_n, l_{-n}] = n l_0 = i n M_{+-}$$

and for the other commutators the embeddings can be checked in the same way. Here we rescaled the metric instead of the generators and as a consequence the algebra (2.37) to which we embed does not change, which is in contrast to the four-dimensional case. Instead now the subalgebras (2.39)-(2.41) change since the metric components depend on n and it is not possible to find embeddings such that both algebras are unchanged. Therefore it does not really matter if we rescale the metric or the generators, the important point is that $\mathcal{B}^3$ contains infinitely many subalgebras which are isomorphic to the Poincaré algebra[3].

[3] With the replacement $M \to nM$ the n drops out of all the commutators in (2.39)-(2.41) and they are therefore isomorphic to the Poincaré algebra with $\eta_{+-} = \eta_{-+} = -\eta_{11} = 1$

Each of these Poincaré subalgebras has its own mass Casimir $\mathcal{C}_n \equiv \eta^{ab} P_a P_b = 2T_{-n}T_{+n} - T_0^2$, which however is not a central element for the entire $\mathcal{B}^3$ (BMS Lie algebra is centreless). Of course, the identification (2.42) - (2.43) is unique up to an automorphism of the full $\mathcal{B}^3$ algebra; e.g. one can apply an involutive automorphism: $l_m \mapsto -l_{-m}, T_m \mapsto -T_{-m}$. We can therefore see that $\mathcal{B}^3$ can be composed entirely from subalgebras which are isomorphic to the Poincaré algebra. Moreover, it turns out that they are maximal finite-dimensional subalgebras of (2.37).

Let us construct explicitly the vacua corresponding to different choices of the Poincaré subgroup. We take as a starting point the general solution of the vacuum Einstein equations in Bondi gauge [28]

$$ds^2 = \Theta(\phi)du^2 - 2dudr + \big(u\Theta'(\phi) + \Xi(\phi)\big)\, dud\phi + r^2 d\phi^2, \tag{2.44}$$

where Θ and Ξ are arbitrary periodic functions of ϕ. In [28] a further generalization is considered by allowing metrics with angular part $r^2 e^{2\varphi} d\phi^2$, $\varphi = \varphi(u, \phi)$ but we consider here only the case $\varphi = 0$. Writing this solution using $z = e^{i\phi}$ one obtains

$$ds^2 = \Theta(z)du^2 - 2dudr + (u\Theta(z)' - i\Xi(z)z^{-1})dudz - \frac{r^2}{z^2}dz^2. \tag{2.45}$$

Plugging the metric components into (2.31) and (2.33) one obtains for the generators of supertranslations

$$T_n = z^n \partial_u - \left(n^2 z^n - \frac{n z^{n+1}}{r} g_{zu}\right) \partial_r + \frac{n z^{n+1}}{r} \partial_z \tag{2.46}$$

and of superrotations

$$l_n = iunz^n \partial_u - i\left(nrz^n + un^3 z^n + \frac{un^2 z^{n+1}}{r} g_{uz}\right) \partial_r + iz^{n+1}\left(1 + \frac{un^2}{r}\right) \partial_z. \tag{2.47}$$

By demanding that the Lie derivative of (2.45) with respect to the generators comprising the embeddings vanishes we obtain a metric that is invariant under the action of these generators. One finds that the only non-vanishing components of the Lie derivative with respect to T_n and l_n are

$$\begin{aligned} \mathcal{L}_{T_n} g_{uu} &= -\frac{n}{r} z^{n+1} \Theta', \quad \mathcal{L}_{T_n} g_{uz} = z^{n-1}(\Theta n + n^3) \\ \mathcal{L}_{l_n} g_{ur} &= -\frac{2un^2 z^{n+1}}{r^2} g_{zu}, \quad \mathcal{L}_{l_n} = 4un^2 z^{n-1} g_{zu}. \end{aligned} \tag{2.48}$$

All Lie derivatives vanish for $n = 0$ and for arbitrary n if we demand $\Xi = \Theta' = 0$ and $\Theta = -n^2$. Notice that for $\Theta = -1$ we correctly recover invariance only under standard Poincaré transformations with $n = 0, \pm 1$. Since the metric is also invariant under the action of T_{-n}, l_{-n} using the same demands on Θ, Ξ we can conclude that the generators

$T_0, T_{\pm n}, l_0, l_{\pm n}$ are the exact Killing vectors of the following solution of Einstein vacuum equations

$$ds^2 = -n^2 du^2 - 2dudr - \frac{r^2}{z^2} dz^2. \tag{2.49}$$

This metric can be diagonalized to become

$$ds^2 = -dt^2 + \frac{1}{n^2}\, dr^2 + r^2 d\phi^2 \,,$$

which after rescaling $r \mapsto r/n$ becomes

$$ds^2 = -dt^2 + dr^2 + r^2 n^2 d\phi^2 \,. \tag{2.50}$$

For $n = 1$ this is the standard flat space metric, but for $n \neq 1$ (2.50) is the metric of the space with conical singularity. This connection between the solutions with a conical defect and the Poincaré subgroups has previously been noted, using a different formulation, in [37].

3 BMS symmetry in the Hamiltonian formulation of general relativity

3.1 Introduction

In their seminal paper from 1974 Regge and Teitelboim [4] have analyzed asymptotically flat spacetimes in the Hamiltonian (ADM) formulation of GR. [4] Their main result was that certain surface integrals at spatial infinity have to be added to the Hamiltonian in order for Hamilton's equations to be well defined. These surface integrals are conserved quantities and form a representation of the Poincaré algebra. They can therefore be understood as the Noether charge associated with the Poincaré symmetry and in particular they asymptotically define the total energy-momentum of the system. Since the considered spacetimes asymptotically approach Minkowski spacetime this appears to be a well-expected and reasonable result. But in light of the above discussed BMS symmetry the question arises why the asymptotic symmetries at null infinity are so different from the ones at spatial infinity. Or are they?

In the recent publication [5] Henneaux and Troessaert revisit the analysis [4] and obtain finite charges generating BMS supertranslations at spatial infinity. Therefore the asymptotic symmetry of asymptotically flat spacetimes, according to [5], is the (unextended) BMS group. This is in contrast to the original result of [4] where these charges were found to vanish and the asymptotic symmetry determined to be the Poincaré group and not the BMS group. The key difference of the two approaches is that Henneaux and Troessaert make the asymptotic expansion in spherical coordinates that makes it possible for them to use different parity conditions on the phase space functions than the ones used by Regge and Teitelboim in their case of Cartesian coordinates expansion. These conditions are a crucial point of the analysis because they guarantee cancellation of divergences, generally plaguing expressions for the asymptotic symplectic form and the charges.

The treatments in [4] and [5] have in common that they take as a starting point a generic asymptotic expansion defining falloff conditions for the spatial metric and its conjugate momenta. Here we instead perform a 3+1 decomposition of a spacetime metric in Bondi coordinates which is asymptotically flat at null infinity, off-shell and fulfills the Bondi gauge except for the determinant condition. We drop the determinant condition since, as will be explained in detail, its presence drastically reduces the asymptotic symmetry at spatial infinity, with spatial (super) translations absent. Using the ADM formalism we express the expansion of the spatial metric and momenta in terms of the metric functions and their derivatives. These expressions are then substituted into the symplectic form, Hamiltonian and diffeomorphism constraint and charges given in [5]. This procedure leads to two interesting insights.

First, it shows that the falloff conditions on the conjugate momenta translate to the

[4] The Hamiltonian formulation referred to here is the one by Arnowitt, Deser and Misner (ADM) [38, 39]. There also exists the covariant phase space formulation [32, 40, 41].

statement that only such spacetimes are allowed that radiate only a finite amount of energy and are therefore physically reasonable. We furthermore find that with the restriction to such spacetimes the falloff conditions are sufficient to remove all divergences in the symplectic structure and there is no need to introduce parity conditions.

Second, we find an asymptotic symmetry at spatial infinity that is larger than the BMS symmetry. The crucial difference to the treatment of [5] is that we do not impose parity conditions which leads to non-vanishing charges which correspond to supertranslations that lie outside of the BMS symmetry. This larger-than-BMS algebra is isomorphic to the one found by Troessaert [17].

The plan of this chapter is as follows. In the next section we review the ADM formalism and recall the results presented in [4] and [5]. In the following section 3.4 we compute the conjugate momenta and express them in terms of the spacetime metric components. We then closely follow the treatment of [5] and analyze which differences arise when substituting these expressions for the momenta. Section 3.5 is devoted to the discussion of the asymptotic symplectic structure, while section 3.6 concerns the Hamiltonian and diffeomorphism constraint. In section 3.7 we discuss the consequences for the asymptotic symmetry.

3.2 Review of ADM formalism and the role of boundary terms

3.2.1 ADM formalism

Following [42] we briefly recall the main features of the ADM formalism, which includes as a crucial ingredient a 3+1 decomposition of spacetime. Such a decomposition is obtained by introducing a foliation of spacetime into spacelike hypersurfaces Σ_t, defined by $t = const.$, where the only condition on t is that the unit normal $n_\alpha \propto \partial_\alpha t$ has to be a future directed timelike vector field. Further, one introduces a time-evolution vector field t^α to define the direction of time derivatives. The defining condition is $t^\alpha \partial_\alpha t = 1$, which allows to interpret the directional derivative $t^\alpha \partial_\alpha$ as ∂_t and thus ensures that the direction of time derivatives is compatible with the meaning of time provided by t.

The vector field t^α is usually decomposed into parts tangential and orthogonal to the spatial hypersurfaces

$$t^\alpha = N n^\alpha + N^a e^\alpha_a, \tag{3.1}$$

where $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ are the tangent vectors on Σ_t, x^α are coordinates in the full spacetime and y^a are coordinates intrinsic to Σ_t. N is referred to as lapse and N^i as shift. Using this decomposition of t^α one can write

$$dx^a = \frac{\partial x^\alpha}{\partial t} dt + \frac{\partial x^\alpha}{\partial y^\alpha} dy^a = t^\alpha dt + e^\alpha_a dy^a \tag{3.2}$$

and for the line element of a generic metric it follows

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -N^2dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt), \tag{3.3}$$

where the definition of the induced metric

$$h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_b \tag{3.4}$$

has been used. The decomposition of the metric (3.3) furthermore implies that

$$\sqrt{-g} = N\sqrt{h}. \tag{3.5}$$

Note, that asymptotic flatness demands for lapse and shift to behave asymptotically as [4, 43]

$$N = 1 + O(1/r),\ N^r = O(1/r),\ N^A = O(1/r^2). \tag{3.6}$$

Using t^α one can define $\dot{h}_{ab} = \pounds_t h_{ab}$ and conjugate momenta

$$\pi^{ab} = \frac{\partial}{\partial \dot{h}_{ab}}\left(\sqrt{-g}\mathcal{L}_G\right), \tag{3.7}$$

where h_{ab} is the induced metric on Σ_t and $\mathcal{L}_G$ is the volume part of the gravitational Lagrangian density defined by the scalar curvature $\mathcal{L}_G = R$[5]. In the 3+1 decomposition this can be expressed as

$$\sqrt{-g}\mathcal{L}_G = \left[{}^3R + \left(h^{ac}h^{bd} - h^{ab}h^{cd}\right)K_{ab}K_{cd}\right]N\sqrt{h}, \tag{3.8}$$

where ${}^3R$ is the scalar curvature associated with h_{ab} and the extrinsic curvature K_{ab} is defined as

$$K_{ab} = \nabla_\beta n_\alpha e^\alpha_a e^\beta_b, \tag{3.9}$$

with ∇_β denoting the covariant derivative with respect to $g_{\mu\nu}$. Now one uses that $\dot{h}_{ab}$ can be written as

$$\dot{h}_{ab} = \mathcal{L}_t h_{ab} = 2NK_{ab} + N_{a|b} + N_{b|a}, \tag{3.10}$$

which can be solved for K_{ab} to calculate the momenta

$$\begin{aligned}\pi^{ab} &= \frac{\partial K_{mn}}{\partial \dot{h}_{ab}}\frac{\partial}{\partial K_{mn}}\left(\sqrt{-g}\mathcal{L}_G\right)\\ &= \sqrt{h}(K^{ab} - Kh^{ab}),\end{aligned} \tag{3.11}$$

[5]We are using units with $G = c = 1$.

where $K = h_{ab}K^{ab}$. Notice that by this definition the momenta are tensor densities of weight 1. Later we are going to use this last expression to write the conjugate momenta in terms of $g_{\alpha\beta}$ and its derivatives.

The gravitational Hamiltonian can finally be expressed as

$$\begin{aligned} H_G &= \int_{\Sigma_t} \left[p^{ab}\dot{h}_{ab} - \sqrt{-g}\mathcal{L}_G \right] d^3y \\ &= \int_{\Sigma_t} \left[N\left(K^{ab}K_{ab} - K^2 - {}^3R \right) - 2N_a \nabla_b \left(K^{ab} - Kh^{ab} \right) \right] \sqrt{h} d^3y \\ &= \int_{\Sigma_t} \left[N\mathcal{H} + N_a \mathcal{H}^a \right] \sqrt{h} d^3y \end{aligned} \tag{3.12}$$

which can equivalently be written in terms of momenta

$$H_G = \int_{\Sigma_t} \left[\frac{N}{\sqrt{h}} \left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2 \right) - {}^3R\sqrt{h}N - 2N_a\nabla_b\pi^{ab} \right] d^3y, \tag{3.13}$$

where ∇_b is the covariant derivative with respect to h_{ab}. One can now use Hamilton's equations to obtain the equations of motion. Using the canonical commutation relations

$$\left\{ h_{ab}(x), \pi^{cd}(y) \right\} = \delta^c_{(a}\delta^d_{b)}\delta(x,y) \tag{3.14}$$

one finds that the equation $\dot{h}_{ab} = \{h_{ab}, H_G\}$ just reproduces (3.10) and for the momenta one finds from $\dot{\pi}^{ab} = \{\pi^{ab}, H_G\}$

$$\begin{aligned} \dot{\pi}^{ab} = &- Nh^{\frac{1}{2}} \left({}^3R^{ab} - \frac{1}{2}h^{ab3}R \right) + \frac{1}{2}Nh^{ab}h^{-\frac{1}{2}} \left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2 \right) \\ &- 2Nh^{-\frac{1}{2}} \left(\pi^{am}\pi^b_m - \frac{1}{2}\pi^{ab}\pi \right) + h^{\frac{1}{2}} \left(\nabla^a\nabla^b N - h^{ab}\nabla^m\nabla_m N \right) \\ &+ \nabla_m \left(\pi^{ab}N^m \right) - \nabla_m N^a \pi^{mb} - \nabla_m N^b \pi^{am}. \end{aligned} \tag{3.15}$$

The momenta conjugate to N, N_a are vanishing, since the Lagrangian density does not contain $\dot{N}, \dot{N}_a$

$$p_N = \frac{\partial \mathcal{L}_G}{\partial \dot{N}} = 0, \quad p_a = \frac{\partial \mathcal{L}_G}{\partial \dot{N}^a} = 0. \tag{3.16}$$

These are referred to as primary constraints [44], because they arise without involving the equations of motion. Preservation of the primary constraints introduces secondary constraints

$$\{p_N, H_G\} = \mathcal{H} = 0, \quad \{p_a, H_G\} = \mathcal{H}_a = 0, \tag{3.17}$$

which implies that the Hamiltonian (3.12) itself is vanishing as a constraint. This is characteristic for a theory that exhibits a gauge symmetry and in the present case reflects the diffeomorphism invariance of GR. To underline this important point we show that the transformations generated by the constraint $\mathcal{H}_a$ are indeed diffeomorphisms. To this end we integrate the constraint over with an arbitrary test function ξ_a

$$\begin{aligned}\Phi[\xi_a] &= \int d^3y \xi_a \mathcal{H}^a \\ &= -2 \int d^3y \xi_a \nabla_b \pi^{ab} \\ &= 2 \int d^3y \nabla_b \xi_a \pi^{ab},\end{aligned} \tag{3.18}$$

where in the last line we have integrated by parts and assumed that boundary terms are vanishing. Transformations are generated via the Poisson bracket

$$\begin{aligned}\delta h_{ab}(y) &= \{g_{ab}(y), \Phi[\xi_k]\} \\ &= 2 \int d^3y' \xi_{k/\ell}(y') \left\{h_{ab}(y), \pi^{k\ell}(y')\right\} \\ &= \nabla_j \xi_i + \nabla_i \xi_j = \mathcal{L}_\xi h_{ab},\end{aligned} \tag{3.19}$$

which we can recognize as the Lie derivative. By its definition the Lie derivative determines how any tensor changes under an infinitesimal diffeomorphism $x^i \to x^i + \xi^i$, which proofs our initial claim.

In the presence of a boundary, i.e. for non-compact spaces, the Hamiltonian can obtain non-vanishing contributions in the form of boundary terms, which we have neglected so far but which will play a crucial role in the following subsection.

3.2.2 The role of boundary terms

To appreciate the role played by boundary terms in the theory we consider again Hamilton's equations, which are defined as the functional derivatives

$$\dot{g}_{ij} = \frac{\delta H}{\delta \pi^{ij}} \tag{3.20}$$

$$\dot{\pi}^{ij} = -\frac{\delta H}{\delta g_{ij}}. \tag{3.21}$$

These are, by definition, the coefficients of δg_{ij} and $\delta \pi^{ij}$ in the variation of the Hamiltonian

$$\delta H = \int d^3x [\mathcal{A}^{ij} \delta g_{ij} + \mathcal{B}_{ij} \delta \pi^{ij}]. \tag{3.22}$$

Therefore, for Hamilton's equations to be defined properly, it is necessary that the variation of the Hamiltonian can be put into the form of (3.22).

The important observation made by Regge and Teitelboim [4] is that in general this is not the case for the Hamiltonian of GR

$$H_G = \int d^3y \left(\xi^\perp \mathcal{H} + \xi^i \mathcal{H}_i\right), \tag{3.23}$$

where $\xi^\perp, \xi^i$ are again arbitrary test function which parametrize the transformations generated by the constraints. Instead they found that extra boundary terms appear

$$\delta H_G = \int d^3y \left[\mathcal{A}^{ij}\delta g_{ij} + \mathcal{B}_{ij}\delta\pi^{ij}\right] + \int d^2\theta K(\xi^\perp, \xi^i), \tag{3.24}$$

which arise from a partial integration, just like the ones we have neglected in (3.18). These boundary terms depend on $\xi^\perp, \xi^i$ so one might be tempted to declare that only such functions are allowed for which the boundary terms vanish. In [4] instead another solution has been proposed that relates the issue of these boundary terms to the asymptotic symmetries of the system.

The first step of this solution is to define the asymptotic behavior of h_{ab} and π^{ab} via boundary conditions, e.g. one demands asymptotic flatness. One then allows only such transformations $\xi^\perp, \xi$ which preserve these boundary conditions. If in the variation of the Hamiltonian no boundary terms appear, everything is well defined and the associated transformation are pure gauge. In the case that the boundary terms are not vanishing it is proposed to redefine the Hamiltonian in the following way. First, one rewrites (3.24) as

$$\delta H_G = \int d^3x \left[\mathcal{A}^{ij}\delta g_{ij} + \mathcal{B}_{ij}\delta\pi^{ij}\right] - \delta B(\xi^\perp, \xi^i) \tag{3.25}$$

and then one defines a new Hamiltonian $H = H_G + B(\xi^\perp, \xi^i)$ such that it has well-defined functional derivatives

$$\delta H = \delta\left(H_G + B(\xi^\perp, \xi^i)\right) = \int d^3x \left[\mathcal{A}^{ij}\delta g_{ij} + \mathcal{B}_{ij}\delta\pi^{ij}\right]. \tag{3.26}$$

In the presence of a boundary therefore something curious happens, namely certain diffeomorphisms seize to be pure gauge transformations since they are generated by a finite Hamiltonian. Such transformations change the system in a way that is in principle measurable and are referred to as improper gauge transformations. The set of transformations that leave the boundary conditions invariant and which are generated by a finite Hamiltonian are then considered to be the physical asymptotic symmetry of the system.

Of course, there is no guarantee that such a procedure always works. The boundary terms that arise could be divergent or it might not be possible to write them as a variation, as is done in (3.25). In the latter case they are referred to as non-integrable. If that is the case one needs to adjust the choice of boundary conditions and repeat the process.

3.3 Summary of previous results

Regge and Teitelboim [4] considered asymptotic flat spaces and defined them using asymptotic Cartesian coordinates via the expansion

$$
\begin{aligned}
g_{ij} &= \delta_{ij} + \frac{1}{r}\bar{h}_{ij} + O\left(r^2\right) \\
\pi^{ij} &= \frac{1}{r^2}\bar{\pi}^{ij} + O\left(r^3\right).
\end{aligned}
\tag{3.27}
$$

They then applied the procedure explained above and found that in order to cancel divergences in the charges the following parity conditions have to be imposed

$$
\bar{h}_{ij}(-\mathbf{n}) = \bar{h}_{ij}(\mathbf{n}), \quad \bar{\pi}_{ij}(-\mathbf{n}) = -\bar{\pi}_{ij}(\mathbf{n}), \tag{3.28}
$$

where $\mathbf{n}$ is the unit vector. The resulting asymptotic symmetry then turns out to be the Poincaré group and the redefined Hamiltonian asymptotically defines the total mass, momentum, angular momentum and boost charge of the system.

An important remark is that in addition to the standard Poincaré translations and Lorentz transformations the boundary conditions are also invariant under angle-dependent translations but the associated charges vanish due to the introduced parity conditions. This means that, according to [4], the asymptotic symmetry algebra is the Poincaré algebra since the symmetry under angle-dependent translations is pure gauge, i.e., the charges associated with these symmetries vanish as a consequence of the chosen parity conditions.

In their paper [5] Henneaux and Troessaert propose different parity conditions which keep Hamilton's equations well-defined but are less strict in the sense that they render the charges associated with angle-dependent translations finite and it is shown that the algebra of these charges is isomorphic to the BMS algebra. Instead of Cartesian coordinates used by Regge and Teitelboim in [5] spherical coordinates are employed and the asymptotic conditions take the form

$$
h_{rr} = 1 + \frac{1}{r}\bar{h}_{rr} + \frac{1}{r^2}h^{(2)}_{rr} + o(r^{-2}) \tag{3.29}
$$

$$
h_{rA} = \bar{h}_{rA} + \frac{1}{r}h^{(2)}_{rA} + o(r^{-1}) \tag{3.30}
$$

$$
h_{AB} = r^2\bar{\gamma}_{AB} + r\bar{h}_{AB} + h^{(2)}_{AB} + o(1) \tag{3.31}
$$

$$
\pi^{rr} = \bar{\pi}^{rr} + \frac{1}{r}\pi^{(2)rr} + o(r^{-1}) \tag{3.32}
$$

$$
\pi^{rA} = \frac{1}{r}\bar{\pi}^{rA} + \frac{1}{r^2}\pi^{(2)rA} + o(r^{-2}) \tag{3.33}
$$

$$
\pi^{AB} = \frac{1}{r^2}\bar{\pi}^{AB} + \frac{1}{r^3}\pi^{(2)AB} + o(r^{-3}) \tag{3.34}
$$

while the parity conditions are

$$
\bar{\lambda} \sim \bar{\pi}^{AB} = \text{even}, \quad \bar{p} \sim \bar{k}_{AB} \sim \bar{\pi}^{rA} = \text{odd}, \tag{3.35}
$$

where

$$\bar{\lambda} = \frac{1}{2}\bar{h}_{rr}, \quad \bar{k}_{AB} = \frac{1}{2}\bar{h}_{AB} + \bar{\lambda}\gamma_{AB} \tag{3.36}$$

$$\bar{p} = 2\left(\bar{\pi}^{rr} - \bar{\pi}^{A}_{A}\right), \quad \pi^{AB}_{(k)} = 2\bar{\pi}^{AB}. \tag{3.37}$$

and γ_{AB} is the unit metric on the sphere. Although a generic expansion of an asymptotically flat metric allows the term $\bar{h}_{rA}$ to be finite it is assumed in [5] that

$$\bar{h}_{rA} = 0 \tag{3.38}$$

which is necessary in order for the boost charges to be integrable.

The parity conditions are introduced to cancel the following logarithmic divergences in the Hamiltonian kinetic term, i.e., the symplectic structure

$$\int dr\frac{1}{r}\int d\theta d\varphi\left(\bar{\pi}^{rr}\dot{\bar{h}}_{rr} + \bar{\pi}^{AB}\dot{\bar{h}}_{AB}\right) = \int dr\frac{1}{r}\int d\theta d\varphi\left(\bar{p}\dot{\bar{\lambda}} + \pi^{AB}_{(k)}\dot{\bar{k}}_{AB}\right), \tag{3.39}$$

which is zero when the parity conditions are assumed because the integral over the sphere of a function with odd parity vanishes. It is furthermore demonstrated that all divergences occurring in the expressions for the charges can be canceled by imposing that the leading order of the Hamiltonian and diffeomorphism constraint is vanishing and no parity conditions have to be involved. The vanishing of the leading order imposes the following relations

$$\bar{\pi}^{rA} = -D_B\bar{\pi}^{BA} \tag{3.40}$$

$$D_A\bar{\pi}^{Ar} = \bar{\pi}^{A}_{A} \tag{3.41}$$

$$D_A D_B k^{AB} = D_A D^A k,, \tag{3.42}$$

which arise from $\mathcal{H}^A, \mathcal{H}^r$ and $\mathcal{H}$ respectively.

To see how the cancellation of the divergences in the boundary terms works consider for instance the divergence proportional to Y^A, which appears in the term $\int d^2xK$ in (3.24) (see [5], Sect. 4 for details of derivation of these terms)

$$\begin{aligned}\int d^2xK_Y &= -2r\oint d^2xY^A\gamma_{AB}\delta\bar{\pi}^{rB} + O(1)\\ &= 2r\oint d^2xY_B D_C\delta\bar{\pi}^{BC} + O(1)\\ &= -2r\oint d^2xD_{(C}Y_{B)}\delta\bar{\pi}^{BC} + O(1),\end{aligned}$$

which vanishes since Y^A are the Killing vectors on the sphere and thus obey

$$D_{(C}Y_{B)} = 0. \tag{3.43}$$

The transformations preserving the above boundary conditions are

$$\xi^\perp = rb + F + O\left(r^{-1}\right), \quad \xi^r = W + O\left(r^{-1}\right), \quad \xi^A = Y^A + \frac{1}{r}I^A + O\left(r^{-2}\right) \tag{3.44}$$

with

$$D_A D_B b + \gamma_{AB} b = 0, \quad \mathcal{L}_Y \gamma_{AB} = 0, \tag{3.45}$$

where b, F, W, Y^A are functions on the sphere and $I^A = \frac{2b}{\sqrt{\gamma}}\bar{\pi}^{rA} + D^A W$. The vectors Y^A describe spatial rotations, b Lorentz boosts, f contains time translations through its zero mode and W contains spatial translations through the $l = 1$ terms in an expansion in spherical harmonics. In order for the parity conditions (3.35) to be preserved as well [5] further assumes

$$F = -3b\bar{\lambda} - \frac{1}{2}b\bar{h} + T, \tag{3.46}$$

where T is an even function on the sphere and W is assumed to be odd. The above defined transformations ξ form under the bracket

$$[\xi_1, \xi_2]_M = [\xi_1, \xi_2]_{SD} + \delta_2^{h,\pi}\xi_1 - \delta_1^{h,\pi}\xi_2 \tag{3.47}$$

the following algebra

$$\widehat{\xi}(\widehat{Y}, \widehat{b}, \widehat{T}, \widehat{W}) = \left[\xi_1\left(Y_1, b_1, T_1, W_1\right), \xi_2\left(Y_2, b_2, T_2, W_2\right)\right]_M, \tag{3.48}$$

where

$$\widehat{Y}^A = Y_1^B \partial_B Y_2^A + \bar{\gamma}^{AB} b_1 \partial_B b_2 - (1 \leftrightarrow 2) \tag{3.49}$$

$$\widehat{b} = Y_1^B \partial_B b_2 - (1 \leftrightarrow 2) \tag{3.50}$$

$$\widehat{T} = Y_1^A \partial_A T_2 - 3b_1 W_2 - \partial_A b_1 D^A W_2 - b_1 D_A D^A W_2 - (1 \leftrightarrow 2) \tag{3.51}$$

$$\widehat{W} = Y_1^A \partial_A W_2 - b_1 T_2 - (1 \leftrightarrow 2), \tag{3.52}$$

which is shown to be isomorphic to the BMS algebra. In (3.47) $[\xi_1, \xi_2]_{SD}$ is the surface deformation bracket defined as

$$[\xi_1, \xi_2]_{SD}^\perp = \xi_1^i \partial_i \xi_2^\perp - \xi_2^i \partial_i \xi_1^\perp \tag{3.53}$$

$$[\xi_1, \xi_2]_{SD}^i = \xi_1^k \partial_k \xi_2^i - \xi_2^k \partial_k \xi_1^i + h^{ik}(\xi_1^\perp \partial_k \xi_2^\perp - \xi_2^\perp \partial_k \xi_1^\perp) \tag{3.54}$$

and $\delta_2^{h,\pi}\xi_1$ is given by

$$\delta_2^{h,\pi}\xi_1 = \frac{\delta \xi_1}{\delta g_{ij}}\delta_{\xi_2} g_{ij} + \frac{\delta \xi_1}{\delta \pi^{ij}}\delta_{\xi_2}\pi^{ij}. \tag{3.55}$$

The terms of the form $\delta_1^{g,\pi}\xi_2$ appear because ξ depends on phase-space functions and one therefore has to take into account the change of ξ induced by the variation of these functions.

The boundary terms found in [5], which correspond to the term $\int d^2xK$ in (3.24), are

$$\begin{aligned}\oint d^2xK_\xi = \delta \oint d^2x &\Big\{-2Y^A\left(\bar{h}_{AB}\bar{\pi}^{rB} + \bar{\gamma}_{AB}\pi^{(2)rB} + \bar{h}_{rA}\bar{\pi}^{rr}\right) \\ &-2\sqrt{\gamma}bk^{(2)} - \sqrt{\bar{\gamma}}\frac{1}{4}b\left(\bar{h}^2 + \bar{h}^{AB}\bar{h}_{AB}\right)\Big\} \\ + \oint d^2x &\Big\{-2I^A\bar{\gamma}_{AB}\delta\bar{\pi}^{rB} - 2W\delta\bar{\pi}^{rr} \\ &-\sqrt{\bar{\gamma}}(2F + \bar{h}b)\delta\left(2\bar{\lambda} + \bar{D}_A\bar{h}_{rA}\right) \\ &+\sqrt{\bar{\gamma}}\left(\bar{h}^{rC}\partial_C b\bar{\gamma}^{AB} - b\bar{D}^A\bar{h}^{rB}\right)\delta\bar{h}_{AB}\Big\} + o\left(r^0\right).\end{aligned} \tag{3.56}$$

It can be seen that the form of F in (3.46) and the assumption $\bar{h}_{rA} = 0$ guarantee integrability of the boundary terms. The expression for the charge, which corresponds to the term B in (3.25), is finally given by

$$\begin{aligned}B_\xi = \oint d^2x &\Big\{Y^A\left(4\bar{k}_{AB}\bar{\pi}^{rB} - 4\bar{\lambda}\gamma_{AB}\bar{\pi}^{rB} + 2\gamma_{AB}\pi^{(2)rB}\right) \\ &+W\bar{p} + T4\sqrt{\gamma}\bar{\lambda} + b\sqrt{\gamma}\left(2k^{(2)} + \bar{k}^2 + \bar{k}^A_B\bar{k}^B_A - 6\overline{\lambda k}\right) \\ &+ b\frac{2}{\sqrt{\gamma}}\gamma_{AB}\bar{\pi}^{rA}\bar{\pi}^{rB}\Big\}\end{aligned} \tag{3.57}$$

and notice in particular that the charges proportional to higher modes of W and T are in general non-vanishing. Supertranslations are therefore part of the asymptotic symmetry for the new set of parity conditions.

Our aim is to express, in a first step, the expansions of the spatial metric and conjugate momenta (3.29)-(3.34) in terms of an asymptotically flat spacetime metric. This will be done by means of the ADM formalism. In a second step we substitute these expressions into the symplectic structure (3.39), constraints (3.40)-(3.42) and charges (3.57) and analyze what are the consequences of this procedure for the asymptotic symmetries.

3.4 Conjugate momenta in terms of spacetime metric components

In this section we are going to perform a 3+1 decomposition of a spacetime metric and use the ADM formalism to express the momenta in terms of components of this metric and their derivatives. We consider a metric that is asymptotically flat at null infinity and is defined by the following expansion

$$
\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu = &-\left(1-\frac{2M}{r}-\frac{\bar{g}_{uu}}{r^2}+O(r^{-3})\right)du^2 \\
&-2\left(1-\frac{\bar{g}_{ur}}{r}-\frac{g^{(2)}_{ur}}{r^2}+O(r^{-3})\right)dudr \\
&+\left(\psi_A+\frac{1}{r}F_A+O(r^{-2})\right)dudx^A \\
&+\left(r^2\gamma_{AB}+rC_{AB}+D_{AB}+O(r^{-1})\right)dx^Adx^B,
\end{aligned} \tag{3.58}
$$

where γ_{AB} is the unit metric on the sphere and all other metric components are functions of (u, x^A). This metric is subject only to the partial Bondi gauge condition

$$
g_{rr}=0, \quad g_{rA}=0 \tag{3.59}
$$

and no further assumptions are made at this stage.

The metric (3.58) is more general than the Bondi metric (2.9), which additionally fulfills the Einstein field equations and is subject to the determinant condition

$$
\det g_{AB}=r^4\det\gamma_{AB}, \tag{3.60}
$$

which implies

$$
\gamma^{AB}C_{AB}=0, \quad \gamma^{AB}D_{AB}=\frac{1}{2}C^{AB}C_{AB}. \tag{3.61}
$$

Here we instead start with an off-shell metric and the field equations will be partially imposed by demanding that the leading order of the Hamiltonian and diffeomorphism constraint has to vanish. We are not imposing the determinant condition since it leads to a metric that is too rigid: in subsection 3.7.2 we are going to demonstrate that it excludes asymptotic translation symmetry.

We choose spacelike hypersurfaces Σ_t to be defined by

$$
t=u+r+f(x^A)+\frac{g(x^A)}{r}=const. \tag{3.62}
$$

which means that f and g parametrize the foliation of spacetime that we choose and in coordinates (t, r, x^A) the metric (3.58) reads

$$
\begin{aligned}
g_{\mu\nu} dx^\mu dx^\nu = & -\left(1 - \frac{2M}{r} - \frac{\bar{g}_{uu}}{r^2} + O(r^{-3})\right) dt^2 \\
& + 2\left(\frac{\bar{g}_{ur} - 2M}{r} + O(r^{-2})\right) dtdr \\
& + 2\left(\partial_A f + \frac{1}{2}\psi_A + O(r^{-1})\right) dtdx^A \\
& + \left(1 + \frac{2M - 2\bar{g}_{ur}}{r} + \frac{\bar{g}_{uu} - 2g^{(2)}_{ur}}{r^2} + O(r^{-3})\right) dr^2 \\
& + \left(-\psi_A + \frac{(4M - 2\bar{g}_{ur})\partial_A f - F_A}{2r} + O(r^{-2})\right) drdx^A \\
& + \left(r^2\gamma_{AB} + rC_{AB} + D_{AB} - \partial_A f \partial_B f - \psi_A \partial_B f\right) dx^A dx^B. \qquad (3.63)
\end{aligned}
$$

By comparing the form of (3.63) with the decomposition (3.3) we find for lapse and shift the following expressions

$$
\begin{aligned}
N &= 1 - \frac{M}{r} + O(r^{-2}), \\
N_r &= \frac{\bar{g}_{ur} - 2M}{r} + O(r^{-2}), \\
N_A &= \partial_A f + \frac{1}{2}\psi_A + O(r^{-1})
\end{aligned}
\qquad (3.64)
$$

and can identify the leading order terms in the metric expansion (3.29)-(3.31) as

$$
\begin{aligned}
&\bar{h}_{rr} = 2M - 2\bar{g}_{ur}, \quad \bar{h}_{rA} = -\frac{\psi_A}{2}, \quad \bar{h}_{AB} = C_{AB} \\
&h^{(2)}_{rr} = \bar{g}_{uu} - 2g^{(2)}_{ur}, \quad h^{(2)}_{rA} = \frac{(4M - 2\bar{g}_{ur})\partial_A f - F_A}{4}, \quad h^{(2)}_{AB} = D_{AB} - \partial_A f \partial_B f - \psi_A \partial_B f.
\end{aligned}
\qquad (3.65)
$$

As they were defined in (3.58) the spacetime metric functions like M are functions of (u, x^A), which might seem a bit odd since they now appear in the components of the induced metric h_{ab} which is described by components (r, x^A). But owing to (3.62) u is not an independent coordinate and neither is t which is now understood as a parameter labeling hypersurfaces. In particular this implies that in the large r limit, in which the expressions (3.65) are defined, we have $u \to -\infty$, which is the defining limit for spatial infinity i^0. This limit for u is from now on implied throughout whenever the spacetime metric functions appear.

The condition (3.38) now takes the form

$$
\psi_A = 0, \qquad (3.66)
$$

and following the arguments of [5] we will eventually also make this assumption. For the sake of obtaining a general form for the expressions of the momenta, however, we will keep ψ_A finite for now and assume that it is vanishing from section 3.6 on.

Notice that the choice $t = u + r + f(x^A) + \frac{g(x^A)}{r}$ is the most general one that is compatible with the fall-off conditions (3.29)-(3.31). Choosing $t = u + rk(x^A) + O(1)$ would lead to

$$h_{ab}dy^a dy^b = dr^2(2k - k^2 + O(r^{-1})) + .. \tag{3.67}$$

which does not agree with the falloff condition (3.29).

The unit normal on the spacelike hypersurfaces is given by

$$n_\alpha = -N\partial_\alpha t = -\left(1 - \frac{M}{r} + O(r^{-2})\right)\partial_\alpha\left(u + r + f(x^A) + \frac{g(x^A)}{r}\right) \tag{3.68}$$

and using this expression to evaluate (3.9) we find the asymptotic expressions for the extrinsic curvature

$$K_{rr} = -\frac{\partial_u M}{r} + O(r^{-2}), \tag{3.69}$$

$$\begin{aligned} K_{rA} = -\frac{1}{4r}\Big(&- 2\partial_A\bar{g}_{ur} + 4\partial_A M - 2\left(\psi_A + 2\partial_A f\right) \\ &- (\psi_B + 2\partial_B f)\gamma^{BC}\partial_u C_{CA} - 4\partial_A f\partial_u M\Big) + O(r^{-2}) \end{aligned} \tag{3.70}$$

$$K_{AB} = \frac{r}{2}\partial_u C_{AB} + O(r^{-2}). \tag{3.71}$$

To illustrate the calculation of the above expressions we give some more details for K_{rr}:

$$K_{rr} = \nabla_u n_u e^u_r e^u_r + \nabla_r n_r e^r_r e^r_r + \nabla_u n_r e^u_r e^r_r + \nabla_r n_u e^r_r e^u_r,$$

where $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ and from (3.62) we have $e^u_r = -1 + O(r^{-2})$ and we also know that $e^r_r = 1$ and $e^A_r = 0$. Evaluating the covariant derivatives we find that the leading order contribution comes from $\nabla_u n_r = \frac{\partial_u M}{r} + O(r^{-2})$ which results in the above expression for K_{rr}. In the same fashion the other components of the extrinsic curvature are obtained.

We have now all the expressions at hand we need to write the momenta in terms of the components of (3.58) and find from evaluating (3.11)

$$\begin{aligned}\pi^{rr} =& -\frac{r}{2}\sqrt{\gamma}\gamma^{AB}\partial_u C_{AB} + \frac{1}{2}\sqrt{\gamma}\big(4\mathcal{M} - \gamma^{AB}\partial_u D_{AB} \\ &+ \gamma^{AB}D_A(2D_B f + \psi_B) + G[\partial_u C_{AB}]\big) + O(r^{-1}) \qquad (3.72)\\ \pi^{rA} =& \frac{1}{4r}\sqrt{\gamma}\Big(- 2\gamma^{AB}D_B\mathcal{M} + 2\gamma^{AB}(2D_B f + \psi_B) \\ &+ G[\partial_u M, \partial_u C_{AB}]\Big) + O(r^{-2}) \qquad (3.73)\\ \pi^{AB} =& \frac{1}{2r}\sqrt{\gamma}\left(2\gamma^{AB}\partial_u M - \left(\gamma^{AB}\gamma^{CD}\partial_u C_{CD} - \partial_u C^{AB}\right)\right) \\ &+ \frac{1}{2r^2}\sqrt{\gamma}\Big[\gamma^{AB}\left(\partial_u \bar{g}_{uu} - 2\mathcal{M}\partial_u \bar{g}_{ur}\right) - (2D^A D^B f + D^{(A}\psi^{B)}) \\ &- \left(\gamma^{AB}\gamma^{CD}\partial_u D_{CD} - \partial_u D^{AB}\right) - \frac{\gamma^{AB}}{2}(\gamma^{CD}(2D_C f + \psi_C)\partial_u \psi_D) \\ &+ \gamma^{CD}(2D_C D_D f + D_C\psi_D)\gamma^{AB} + G[\partial_u M, \partial_u C_{AB}]\Big] + O(r^{-3}), \qquad (3.74)\end{aligned}$$

where

$$\mathcal{M} = \bar{g}_{ur} - 2M \qquad (3.75)$$

and G stands for lengthy terms proportional to either $\partial_u C_{AB}$ or $\partial_u M$.

It might seem strange that derivatives over u appear in the above expressions for the extrinsic curvature and conjugate momenta since they live on a spacelike surface with coordinates (r, x^A). But using (3.62) we could write ∂_u in terms of derivatives over (t, r, x^A) and setting in the resulting expressions $t = const.$ we find that the curvature and momenta are described solely in terms of (r, x^A), as they should.

Comparing the expressions (3.72)-(3.74) with the falloff conditions (3.32)-(3.34) we see that the $O(r)$ term of π^{rr} and the $O(r^{-1})$ term of π^{AB} should be vanishing. Thus we find that the falloff conditions translate into the following conditions on the spacetime metric functions near spatial infinity, i.e. in the limit $u \to -\infty$

$$\partial_u C_{AB} \sim u^{-(1+\varepsilon)}, \quad \partial_u M \sim u^{-(1+\varepsilon)}, \quad \varepsilon > 0, \qquad (3.76)$$

which provide an extra damping factor so that we can neglect all terms proportional to them. We therefore only consider such spacetimes that satisfy this condition. As has been outlined in section 2.1.1 the above derivatives over u are expressing on-shell the rate of gravitational radiation and the conditions (3.76) can therefore be interpreted such that the total amount of energy radiated by the system must be finite. This requirement to only allow such "physically reasonable" spacetimes has already been pointed out in [4](therein after eq. (2.1)).

3.5 Finiteness of symplectic structure

The falloff conditions defined in (3.29)-(3.34) are not sufficient to remove divergences in the symplectic structure

$$\int d^3x \pi^{ab} \dot{h}_{ab}, \tag{3.77}$$

since terms of order $O(r^{-1})$ appear which are logarithmically divergent in the large r limit which we consider. As was explained above, after eq. (3.39), the authors of [5] remove these divergences by introducing parity conditions on the leading order terms in the expansion of the metric and momenta. The terms which are potentially divergent are the following ones

$$\int dr \frac{1}{r} \int d\phi d\theta \left(\bar{\pi}^{rr} \dot{\bar{h}}_{rr} + \bar{\pi}^{AB} \dot{\bar{h}}_{AB} + \bar{\pi}^{rA} \dot{\bar{h}}_{rA} \right) \tag{3.78}$$

and we could now impose the parity conditions of [5] and investigate what they imply for the spacetime metric functions and, since the momenta depend on $f(x^A)$, for the foliation of spacetime.

But using the definition of $\dot{h}_{ab}$ (3.10) to express it in terms of the spacetime metric functions we obtain

$$\dot{\bar{h}}_{rr} = -2\partial_u M, \quad \dot{\bar{h}}_{rA} = 0, \quad \dot{\bar{h}}_{AB} = \partial_u C_{AB}, \tag{3.79}$$

and we therefore find that the condition (3.76) that we imposed on allowed spacetimes provides an additional damping and is therefore sufficient, together with the falloff conditions, to make the asymptotic symplectic structure finite. We can therefore conclude that for the class of spacetimes that we consider no parity conditions are needed to cancel divergences in the symplectic structure.

3.6 Leading order of constraints

In [5] the leading order of the Hamiltonian and diffeomorphism constraint is assumed to vanish in order to cancel divergences in the asymptotic charges, as was explained in subsection 3.3. In this section we show which restrictions on the form of the momenta this requirement implies. We are also going to assume $\psi_A = 0$ from now on.

Substituting (3.73) into (3.41)

$$\frac{\sqrt{\gamma}}{2} \gamma^{AB} D_B (2f - \mathcal{M}) = -D_B \bar{\pi}^{AB} \tag{3.80}$$

which implies

$$\frac{\sqrt{\gamma}}{2} \gamma^{AB} (2f - \mathcal{M}) = -\bar{\pi}^{AB} + A\gamma^{AB} \sqrt{\gamma}\,, \tag{3.81}$$

with A being an arbitrary constant. Plugging in (3.74) this equation expresses a relation between several spacetime metric functions at spatial infinity

$$\begin{aligned}\frac{\gamma^{AB}}{2}(2f-\mathcal{M}) =&\gamma^{AB}(\partial_u\bar{g}_{uu}-2\mathcal{M}\partial_u\bar{g}_{ur}-\gamma^{CD}\partial_u D_{CD}+2D^2 f)\\ &+\partial_u D^{AB}-2D^A D^B f.\end{aligned} \tag{3.82}$$

Substituting (3.81) into (3.41) yields

$$(D^2+2)(2f-\mathcal{M})=A \tag{3.83}$$

and upon expanding $2f-\mathcal{M}$ in spherical harmonics Y_{lm} satisfying $D^2Y_{lm}=-l(l+1)Y_{lm}$ we find that this equation has a general solution of the form

$$2f-\mathcal{M}=A+\sum_{m=-1}^{1}a_m Y_{1m} \tag{3.84}$$

with a_m being arbitrary constants.

To summarize, the momenta are now expressed in terms of the spacetime metric functions as

$$\bar{\pi}^{rr} =\frac{\sqrt{\gamma}}{2}(4\mathcal{M}+2D^2 f-\gamma^{AB}\partial_u D_{AB}), \tag{3.85}$$

$$\bar{\pi}^{rA} =\frac{\sqrt{\gamma}}{2}\gamma^{AB}D_B(2f-\mathcal{M}), \tag{3.86}$$

$$\bar{\pi}^{AB} =\frac{\sqrt{\gamma}}{4}\gamma^{AB}(2\mathcal{M}-4f+A), \tag{3.87}$$

subject to the conditions (3.82) and (3.84).

3.7 Asymptotic symmetries

In this section we discuss the asymptotic symmetries of h_{ab} and π^{ab}, in particular we will analyze which transformations preserve the falloff conditions (3.29)-(3.34) and the gauge condition (3.38). This will reproduce the expressions (3.44) and (3.45) which were already given in [5]. We are giving here the details of this derivation to stress the fact that, as we are going to show, the preservation of the falloff conditions allows for a large group of supertranslations parametrized by two arbitrary functions on the sphere. These details are furthermore needed for our discussion of the determinant condition (3.60) in subsection 3.7.2.

To discuss the asymptotic symmetries we will evaluate the change in the canonical variables generated by the Hamiltonian $H=\int d^3x(\xi^{\perp}\mathcal{H}+\xi^a\mathcal{H}_a)$ which is given by, see

[39]

$$\delta_\xi h_{ab} = 2\xi^\perp h^{-1/2}\left(\pi_{ab} - \frac{1}{2}h_{ab}\pi\right) + \mathcal{L}_\xi h_{ab}. \tag{3.88}$$

$$\begin{aligned}\delta\pi^{ab} =& -\xi^\perp h^{\frac{1}{2}}\left(R^{ab} - \frac{1}{2}h^{ab}R\right) + \frac{1}{2}\xi^\perp h^{ab}h^{-\frac{1}{2}}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right)\\ &- 2\xi^\perp h^{-\frac{1}{2}}\left(\pi^{am}\pi^b_m - \frac{1}{2}\pi^{ab}\pi\right) + h^{\frac{1}{2}}\left(\xi^{\perp|ab} - h^{ab}\xi^{\perp|m}{}_{|m}\right)\\ &+ \left(\pi^{ab}\xi^m\right)_{|m} - \xi^a_{|m}\pi^{mb} - \xi^b{}_{|m}\pi^{am},\end{aligned} \tag{3.89}$$

where $\mathcal{L}_\xi h_{ab}$ is the Lie derivative

$$\mathcal{L}_\xi h_{ab} = \xi_{a|b} + \xi_{b|a}. \tag{3.90}$$

3.7.1 Preservation of falloff and gauge conditions

From the preservation of the falloff conditions (3.29)-(3.31) we obtain the demands

$$\delta h_{rr} = O\left(r^{-1}\right), \quad \delta h_{rA} = O(1), \quad \delta h_{AB} = O(r) \tag{3.91}$$

and we now want to find such $\xi^\perp, \xi^a$ that the change of h_{ab} defined in (3.88) obeys them.

Using the expansion of Christoffel symbols associated with h_{ab}

$$\Gamma^r_{AB} = -r\bar{\gamma}_{AB} + O(1) \tag{3.92}$$
$$\Gamma^A_{BC} = \bar{\Gamma}^A_{BC} + O(r^{-1}) \tag{3.93}$$
$$\Gamma^r_{rA} = \frac{1}{2r}\left(\partial_A\bar{h}_{rr} + \psi_A\right) + O(r^{-2}) \tag{3.94}$$
$$\Gamma^r_{rr} = -\frac{\bar{h}_{rr}}{2r^2} + O(r^{-3}) \tag{3.95}$$
$$\Gamma^A_{rB} = \frac{1}{r}\delta^A_B + O(r^{-2}) \tag{3.96}$$
$$\Gamma^A_{rr} = -\frac{\bar{\gamma}^{AB}\partial_B\bar{h}_{rr}}{2r^3} + O(r^{-3}), \tag{3.97}$$

we find the following transformation of h_{rr}

$$\delta h_{rr} = \frac{2\xi^\perp}{\sqrt{\gamma}r^2}\left(\bar{\pi}_{rr} - \frac{1}{2}\bar{h}_{rr}\bar{\pi}\right) + \xi^A\partial_A\frac{\bar{h}_{rr}}{r} + 2\partial_r\xi^r - 2\psi_A\partial_r\xi^A + (\text{subleading}), \tag{3.98}$$

where

$$\bar{\pi} = \bar{\pi}^{rr} + \gamma_{AB}\bar{\pi}^{AB}. \tag{3.99}$$

Comparing this with (3.91) we find that a large r expansion of $\xi^\perp, \xi^a$ has to be of the form

$$\begin{aligned}
\xi^\perp &= rb + F + O(r^{-1}), \\
\xi^A &= Y^A + \frac{1}{r} I^A + O(r^{-2}), \\
\xi^r &= W + \frac{1}{r}\xi^r_1 + O(r^{-2}),
\end{aligned} \tag{3.100}$$

where all terms in this expansion are functions on the sphere.

The transformation of h_{rA}

$$\begin{aligned}
\delta h_{rA} =& \frac{2b}{\sqrt{\gamma}}\bar{\pi}_{rA} - \frac{1}{2} Y^B D_B \psi_A + \partial_A W \\
& - \frac{1}{2}\psi_B D_A Y_0^B - \gamma_{AB} I^B + \text{(subleading)}
\end{aligned} \tag{3.101}$$

does not give any further restrictions on $\xi^\perp, \xi^a$.

To fulfill the demand that δh_{AB}

$$\begin{aligned}
\delta h_{AB} =& r^2 D_{(A} Y_{B)} \\
& + r\Big(2W\gamma_{AB} + Y^C D_C C_{AB} + \psi_{(B}\gamma_{A)C} Y^C \\
& + 2C_{C(B} D_{A)} Y^C + 2\gamma_{C(B} D_{A)} I^C + \frac{2b}{\sqrt{\gamma}}\left(\bar{\pi}_{AB} - \frac{1}{2}\gamma_{AB}\bar{\pi}\right)
\end{aligned} \tag{3.102}$$

is of order $O(r)$ we have to assume that Y^B are the Killing vectors on the 2-sphere

$$D_{(A} Y_{B)} = 0. \tag{3.103}$$

So far there are no restrictions on b but it is fixed from the preservation of the asymptotic form of momenta which demands

$$\delta \pi^{rr} = O(1), \quad \delta \pi^{rA} = O(r^{-1}), \quad \delta \pi^{AB} = O(r^{-2}). \tag{3.104}$$

One can check that there are no new restrictions coming from the first two conditions in (3.104). The third one, however, does lead to a new restriction and reads

$$\begin{aligned}
\delta \pi^{AB} &= r^2 \sqrt{\gamma}\left(\nabla^A \nabla^B \xi^\perp - h^{AB}\nabla^i \nabla_i \xi^\perp\right) + \text{(subleading)} \\
&= \frac{\sqrt{\gamma}}{r^2}\left(\gamma^{AC}\gamma^{AD}\nabla_C \nabla_D \xi^\perp - \gamma^{AB}\gamma^{CD}\nabla_C \nabla_D \xi^\perp\right) + \text{(subleading)}.
\end{aligned} \tag{3.105}$$

Now consider(we use the notation $\nabla_A = D_A +$ (subleading))

$$\begin{aligned}\nabla_C\nabla_D\xi^\perp &= r\partial_C\partial_D b - \Gamma^r_{CD}\partial_r(rb) - r\Gamma^A_{CD}\partial_A b + O(1)\\ &= rD_C D_B b + r\gamma_{CD}b,\end{aligned} \tag{3.106}$$

where we used (3.92) and plugging this back into the previous equation we obtain

$$\delta\pi^{AB} = \frac{1}{r}\left(D^A D^B b - \gamma^{AB}D^2 b - \gamma^{AB}b\right) + O(r^{-2}). \tag{3.107}$$

Vanishing of the leading order therefore imposes the condition

$$D_A D_B b - \gamma_{AB}D^2 b - \gamma_{AB}b = 0 \Rightarrow D^2 b = -2b \tag{3.108}$$

and therefore we find that b has to satisfy

$$D_A D_B b + \gamma_{AB} b = 0\,, \tag{3.109}$$

whose only solution is b being a linear combination of three $l = 1$ harmonics with constant coefficient.

Now we consider the preservation of the gauge condition $\bar{h}_{rA} = 0$. From (3.101) one can directly see that in order to have $\delta\bar{h}_{rA} = 0$ we need to assume

$$I^A = D^A W + \frac{2b}{\sqrt{\gamma}}\bar{\pi}^{rA}, \tag{3.110}$$

which means that the preservation of the gauge choice

$$\bar{h}_{rA} = -\frac{1}{2}\psi_A = 0, \tag{3.111}$$

determines the subleading term of ξ^A.

To summarize, F and W are not constrained by the boundary conditions and are associated with angle-dependent translations, temporal and spatial ones, respectively. Y^A are the three Killing vectors on the sphere parametrizing rotations and b contains only $l = 1$ harmonics and parametrizes three boosts. Except for the assumptions of parity on W and F we have therefore reproduced the asymptotic symmetry transformations given in [5] and the corresponding surface terms are therefore identical with (3.56), which has been derived for F, W having no definite parity. Integrability of these surface terms demands that F is of the form $F = -\frac{1}{2}\bar{h}b + T(x^A)$, where $T(x^A)$ is a general function on the sphere. Any function of $\bar{\lambda}$ could be added to F without spoiling integrability, which introduces an ambiguity in the expression of the charges. We choose for F the form (3.46).

In [5] it is shown that transformations with W=odd, T=even form an algebra isomorphic to the BMS algebra found at null infinity. The odd and even functions combine to give the single function parametrizing supertranslations at null infinity. Since we do not involve any parity conditions it appears that the resulting asymptotic symmetry is larger than the BMS symmetry, as long as the associated charges are finite. Before proceeding with the discussion of the charges, however, we turn to the aforementioned determinant condition.

3.7.2 Determinant condition and reduction of symmetry

In the previous subsection we have found that a large group of supertranslations and Lorentz transformations preserve the falloff conditions (3.29)-(3.34). We are now going to demonstrate that the determinant condition (3.60) is not preserved by spatial (super) translations.

The determinant condition implies, see (2.5)-(2.7)

$$\gamma^{AB}C_{AB} = 0 \tag{3.112}$$

and transformations preserving this condition must fulfill

$$\delta(\gamma^{AB}C_{AB}) = \gamma^{AB}\delta\bar{h}_{AB} = 0, \tag{3.113}$$

where we have used that $\delta\gamma_{AB} = 0$ and $C_{AB} = \bar{h}_{AB}$. Substituting (3.102) we find

$$\gamma^{AB}\delta\bar{h}_{AB} = 2(D^2+2)W + \frac{4}{\sqrt{\gamma}}D_A(b\bar{\pi}^{rA}) - \frac{2b}{\sqrt{\gamma}}\bar{\pi}^{rr} = 0 \tag{3.114}$$

and using the expressions for the momenta (3.85) and (3.86) we obtain

$$2(D^2+2)W = -2D^A(bD_A(2f-M)) + b(4\mathcal{M} + D^2 f - \gamma^{AB}\partial_u D_{AB}). \tag{3.115}$$

This equation has no solution for W since the RHS is in general non-vanishing and inevitably contains $l = 1$ harmonics[6] which can not be produced by the LHS.

We can therefore conclude that spatial translations, i.e. standard Poincaré ones and supertranslations, do not preserve the condition (3.112). Imposing this condition would therefore drastically reduce the asymptotic symmetries.

3.7.3 Discussion of charges

In the previous subsections we found that the asymptotic symmetry appears to be larger than the BMS symmetry at null infinity because there are two arbitrary functions on the sphere parametrizing translations. In the treatment of [5] the charges associated with transformations outside of BMS are vanishing due to the imposed parity conditions and these transformations are therefore pure gauge. Here we show that in our treatment all the charges associated with T and W are in fact non-vanishing. We will also check that the boost and rotational charges are all non-vanishing.

[6]In fact b purely consists of $l = 1$ harmonics and $(2f - \mathcal{M})$ of $l = 0, 1$ harmonics.

To obtain the form of the charges in terms of the spacetime metric functions we can use the expression (3.57) because it has been derived for general T, W, which corresponds to the case that we are considering. We then only need to substitute the expressions for h_{ab} and π^{ab}, which we have already obtained from the 3+1 decomposition. From (3.57) the charges associated with translations are given as

$$\mathcal{B}_{T,W} = \oint d^2x\{2T\sqrt{\gamma}\bar{h}_{rr} + 2W\left(\bar{\pi}^{rr} - \bar{\pi}^A_A\right)\}, \tag{3.116}$$

the boost charges are

$$\mathcal{B}_b = \oint d^2x\left[b\sqrt{\gamma}\left(2k^{(2)} + \bar{k}^2 + \bar{k}^A_B\bar{k}^B_A - 6\overline{\lambda k}\right) + b\frac{2}{\sqrt{\gamma}}\gamma_{AB}\bar{\pi}^{rA}\bar{\pi}^{rB}\right] \tag{3.117}$$

and finally the rotational charges

$$\mathcal{B}_Y = \oint d^2xY^A\left(4\bar{k}_{AB}\bar{\pi}^{rB} - 4\bar{\lambda}\gamma_{AB}\bar{\pi}^{rB} + 2\gamma_{AB}\pi^{(2)rB}\right), \tag{3.118}$$

where $k^{(2)}$ is defined via the expansion

$$K^A_B = h^{AC}K_{BC} = -\frac{1}{r}\delta^A_B + \frac{1}{r^2}\bar{k}^A_B + \frac{1}{r^3}k^{(2)A}_B + O\left(r^{-3}\right) \tag{3.119}$$

$$K_{AB} = \frac{1}{2\lambda}\left(-\partial_r h_{AB} + \nabla_A h_{rB} + \nabla_B h_{rA}\right) \tag{3.120}$$

$$\lambda = \frac{1}{\sqrt{h^{rr}}}. \tag{3.121}$$

Substituting the expressions for the momenta (3.85) and (3.87) into (3.116) yields for the translational charge

$$\mathcal{B}_{W,T} = \oint d^2x\sqrt{\gamma}\{T2\bar{h}_{rr} + \frac{W}{2}(2\mathcal{M} + 2(D^2+2)f - \gamma^{AB}\partial_u D_{AB} - A)\} \tag{3.122}$$

and using the condition (3.84) we can eliminate f and A from this expression and obtain

$$\mathcal{B}_{W,T} = \oint d^2x\sqrt{\gamma}\{T2\bar{h}_{rr} + \frac{W}{2}((D^2+4)\mathcal{M} - \gamma^{AB}\partial_u D_{AB})\}. \tag{3.123}$$

As a quick sanity check we calculate the ADM mass, obtained from the above expressions for $T = 1, W = 0$, for the outgoing Vaidya spacetime

$$ds^2 = -\left(1 - \frac{2m(u)}{r}\right)du^2 - 2dudr + r^2d\Omega^2 \tag{3.124}$$

which can be thought of as a radiating generalization of Schwarzschild spacetime. It is a solution of Einstein equations with energy-momentum $T_{uu} = -\frac{dm/du}{4\pi r^2}$. The ADM mass of the Vaidya spacetime according to (3.123) is

$$m_{ADM} = 16\pi \lim_{u\to-\infty} m(u), \tag{3.125}$$

which is indeed the correct expression, see for example chapter 4.3.5 in [42](which agrees up to the normalization factor of 16π).

To investigate which modes of T, W lead to finite charges we expand them and the metric functions in spherical harmonics and use their orthonormality and the fact that any spherical harmonic by itself vanishes when integrated over the sphere. This implies that the only finite terms are the ones where each factor has a contribution from the same mode. If $\bar{h}_{rr}$, for instance, was a constant then the first term in the above charge would only have a non-vanishing contribution from the zero mode of T. We can therefore see that there are finite contributions from all modes of T and W present, since only the combination $f - 2\mathcal{M}$ is constrained to $l = 0$ and $l = 1$ modes but $\mathcal{M}$ itself contains in general contributions from all modes and so does $\bar{h}_{rr}$.

Next we are going to write the rotation and boost charges in terms of the spacetime metric functions and in doing so we will specialize to the case $f = const.$ and $g = const.$. We make this choice since otherwise these expressions are very lengthy and it is justified because it does not affect whether the charges are vanishing or not[7], which is the only thing that we want to check. For $f = const., g = const.$ we find for the subleading contribution to π^{rA}

$$\begin{aligned}\pi^{(2)rA} =& \frac{\sqrt{\gamma}}{4}\Big(D^A\mathcal{M}\gamma^{AB}C_{AB} - 4C^{AB}D_B\mathcal{M} + 3F^A \\ &- 2D^A g^{(2)}_{ur} + 2D^A\bar{g}_{uu} - 2D^A\mathcal{M}\partial_u\bar{g}_{uu} - 4MD^A\bar{g}_{ur} \\ &+ 16MD^A M - 2\bar{g}_{ur}D^A\bar{g}_{ur}\Big) \end{aligned}\tag{3.126}$$

$$\tag{3.127}$$

and together with

$$4\bar{k}_{AB}\bar{\pi}^{rB} - 4\bar{\lambda} = -2C_{AB}D^B\mathcal{M} \tag{3.128}$$

we therefore find that the rotational charge (3.129)

$$\mathcal{B}_Y = \oint d^2x Y^A \left(-2C_{AB}D^B\mathcal{M} + 2\gamma_{AB}\pi^{(2)rB}\right) \tag{3.129}$$

is indeed non-vanishing for all modes of Y^A since F^A and $\mathcal{M}$ contain in general contributions from all modes of spherical harmonics. Allowing for general f and g would not change this conclusion(even if they do not drop out) in particular because there is no condition that would relate F_A to either of these functions.

[7] In fact one would expect that f and g drop out of the expressions for the boost and rotational charges since they should be independent of the foliation just as they dropped out of the supertranslation charge. However, since these expressions are for general f and g rather complicated we have not been able to show this.

To obtain the expression for the boost charge (3.117) we first evaluate (3.120) and (3.121) using (3.65) and find

$$K_{AB} = \frac{1}{2\lambda}\left(-2r\gamma_{AB} - C_{AB} - \frac{1}{2r}D_{(A}F_{B)}\right) + O(r^{-4}) \tag{3.130}$$

and

$$\frac{1}{\lambda} = 1 - \frac{\bar{h}_{rr}}{r} + \frac{L}{r^2} + O(r^{-3}), \tag{3.131}$$

where

$$L = -4\bar{g}_{uu} + 12(\mathcal{M}\bar{g}_{ur} + M^2) + 8g^{(2)}_{ur}. \tag{3.132}$$

To calculate $k^{(2)}$ from (3.119) we also need

$$h^{AB} = \frac{1}{r^2}\gamma^{AB} - \frac{1}{r^3}C^{AB} + \frac{1}{r^4}\left(C^A_D C^{DB} - D^{AB}\right) + O(r^{-5}) \tag{3.133}$$

and find

$$k^{(2)} = 2\gamma_{AB}D^{AB} - C_{AB}C^{AB} - 4L - D_A F^A \tag{3.134}$$

and finally obtain for the boost charge

$$\begin{aligned}\mathcal{B}_b = \oint d^2x\sqrt{\gamma}b\Big(&D_A\mathcal{M}D^A\mathcal{M} + 2\gamma_{AB}D^{AB} - 4L - D_A F^A \\ &+ \frac{1}{4}(\gamma_{AB}C^{AB})^2 - \frac{3}{4}C_{AB}C^{AB} - \frac{3}{2}\bar{h}^2_{rr} - \frac{5}{2}\gamma_{AB}C^{AB}\bar{h}_{rr}\Big).\end{aligned} \tag{3.135}$$

Again we can see that the charge is finite for all modes of b since $F^A, \mathcal{M}$ and $\bar{h}_{rr}$ contain contributions from all modes.

3.7.4 Discussion of the asymptotic symmetry

From our discussion of the charges it has become clear that the asymptotic symmetry we find at spatial infinity is larger than the BMS symmetry. The crucial difference to the results of [5] is that the charges associated with even W and odd T are non-vanishing.

A larger-than-BMS asymptotic symmetry at spatial infinity has been found previously by Ashtekar and Hansen [24]. The Spi algebra they find has the same structure as BMS, namely a semi-direct product of the abelian ideal of supertranslations and the Lorentz algebra. The difference lies in the size of the supertranslation ideal, which for the BMS algebra corresponds to functions on the 2-sphere whereas for Spi it corresponds to functions on the three-dimensional hyperboloid.

Also Troessaert [17] finds an asymptotic symmetry at spatial infinity which is larger than BMS, but smaller than the Spi algebra. By additionally assuming that the spacetime metric considered therein has to be asymptotically flat not only at spatial infinity but also at null infinity conditions on the metric functions are found which reduce the algebra to one that is isomorphic to the BMS algebra. This algebra is then in turn shown by [5] to be isomorphic to the algebra (3.48) with odd W and even T. A fact that has not been stressed therein, although it is implicit, is that the algebra (3.48) with arbitrary W and T is isomorphic to the one found by [17] before cutting it down to BMS. Following the explanations in the appendix of [5] this can be seen as follows.

Troessaert [17] finds the following asymptotic algebra

$$\left[\left(\mathcal{Y}_1,\omega_1'\right),\left(\mathcal{Y}_2,\omega_2'\right)\right]=\left(\left[\mathcal{Y}_1,\mathcal{Y}_2\right],\mathcal{Y}_1^a\partial_a\omega_2'-\frac{s}{2}\psi_1\omega_2'-(1\leftrightarrow 2)\right)=\left(\widehat{\mathcal{Y}},\widehat{\omega}'\right), \tag{3.136}$$

where $x^a=(s,x^A)$ are coordinates on the unit hyperboloid, $\mathcal{Y}^a$ represents the Lorentz algebra, $\omega'=\sqrt{1-s^2}\omega$ and $\omega(x^a)$ parametrize a sub-set of Spi supertranslations [24]. Full Spi supertranslations would be given by general functions $\omega(x^a)$ but, as [17] explains, to remove divergences in the symplectic structure one has to demand

$$(D^aD_a+3)\omega=0. \tag{3.137}$$

The general solution of this equation is shown to be of the form

$$\begin{aligned}\omega&=\frac{1}{\sqrt{1-s^2}}\left(\widehat{\omega}^{\text{even}}+\widehat{\omega}^{odd}\right)\\ \widehat{\omega}^{\text{even}}&=\sum_{l,m}\widehat{\omega}_{lm}^{V}V_l(s)Y_{lm}^0\left(x^A\right),\quad \widehat{\omega}^{\text{odd}}=\sum_{l,m}\widehat{\omega}_{lm}^{W}W_l(s)Y_{lm}^0\left(x^A\right),\end{aligned} \tag{3.138}$$

where odd and even refers to the combination of time reversal $s\to -s$ and antipodal mapping $x^A\to -x^A$ and $V_l(s),W_l(s)$ are defined in terms of Legendre polynomials and Legendre functions of the second kind.

Rotations are parametrized by Killing vectors on the 2-sphere $\mathcal{Y}_R^A(x^A)$

$$\mathcal{Y}^s=0,\quad \mathcal{Y}^A=\mathcal{Y}_R^A \tag{3.139}$$

and boosts by $\psi(x^A)$ such that $D^2\psi+2\psi=0$

$$\mathcal{Y}^s=-\frac{1}{2}\left(1-s^2\right)\psi,\quad \mathcal{Y}^A=-\frac{1}{2}s\gamma^{AB}\partial_B\psi. \tag{3.140}$$

One can check that the Lorentz algebras in (3.136) and (3.48) are isomorphic under the identification $\mathcal{Y}_R^A=Y^A$ and $\psi=2b$. The action of the Lorentz algebra on ω' in (3.136) can then be written as

$$\widehat{\omega}'=Y_1^A\partial_A\omega_2'-sb_1\omega_2'-s\partial^Ab_1\partial_A\omega_2'-\left(1-s^2\right)b_1\partial_s\omega_2'-(1\leftrightarrow 2). \tag{3.141}$$

The connection with the form of the algebra in the ADM description in (3.48) can be made by defining W, T as initial conditions at $s=0$

$$\omega|_{s=0} = \omega'\big|_{s=0} = W\left(x^A\right), \quad \partial_s\omega|_{s=0} = \partial_s\omega'\big|_{s=0} = T\left(x^A\right). \tag{3.142}$$

One can check from the definitions of $V_l(s)$ and $W_l(s)$ that $\omega|_{s=0}$ and $\partial_s\omega|_{s=0}$ contain contributions from all modes of spherical harmonics and that therefore W and T such defined are arbitrary functions on the sphere. Substituting these definitions in (3.141) one obtains

$$\widehat{W} = Y_1^A\partial_A W_2 - b_1 T_2 - (1 \leftrightarrow 2), \tag{3.143}$$

which is equal to the corresponding expression in (3.48).

Acting with the s derivative on (3.141) yields

$$\begin{aligned}\partial_s\widehat{\omega}' =& Y_1^A\partial_A\partial_s\omega_2' - b_1\omega_2' - sb_1\partial_s\omega_2' - \partial^A b_1\partial_A\omega_2' - s\partial^A b_1\partial_A\omega_2' \\ &+ 2sb_1\partial_s\omega_2' - (1-s^2)b_1\partial_s^2\omega_2' - (1 \leftrightarrow 2).\end{aligned} \tag{3.144}$$

The expression for $\partial_s^2\omega'\big|_{s=0}$ can be obtained from

$$\left(\mathcal{D}_a\mathcal{D}^a + 3\right)\omega = -\left(1-s^2\right)^2\partial_s^2\omega + \left(1-s^2\right)D^2\omega + 3\omega = 0 \tag{3.145}$$

and first one has

$$\partial_s^2\omega\Big|_{s=0} = D^2\omega\Big|_{s=0} + 3\omega|_{s=0} \tag{3.146}$$

and together with

$$\partial_s^2\omega'\Big|_{s=0} = -\omega|_{s=0} + \partial_s^2\omega\Big|_{s=0} \tag{3.147}$$

one finally has

$$\partial_s^2\omega'\Big|_{s=0} = 2\omega|_{s=0} + D^2\omega\Big|_{s=0}. \tag{3.148}$$

Substituting this and the above definitions in (3.144) and evaluating at $s=0$ one finally obtains

$$\widehat{T} = Y_1^A\partial_A T_2 - 3b_1 W_2 - \partial_A b_1 D^A W_2 - b_1 D^2 W_2 - (1 \leftrightarrow 2), \tag{3.149}$$

which is exactly of the form given in (3.48).

To summarize, since we do not impose parity conditions we find an asymptotic symmetry at spatial infinity that is larger than the BMS algebra but smaller than the Spi algebra. Our result would therefore suggest that the tension arising from the presence of different asymptotic symmetries at spatial infinity and null infinity still remains.

4 Gauging BMS

4.1 Introduction

This section is concerned with the relation of GR and gauge theories. That there is such a relation is not obvious in the usual coordinate based description but it becomes apparent in the so-called vielbein formulation, which uses basis vectors which are not obtained from a coordinate system. Using this formulation one can for instance show that the curvature, i.e. the Riemann tensor, takes the form of the field strength tensor known from gauge theories. Since this formalism is the basis for everything that follows in this section we will shortly review it in subsection 4.2.

It was shown by Witten [6], see also [45], that in 2+1 dimensions GR is equivalent to a gauge theory with the Poincaré group as gauge group and with a pure Chern-Simons action. Such a gauge theory is essentially like an ordinary Yang-Mills theory, but with an action of the form $\int AdA + A \wedge A \wedge A$ instead of the Yang-Mills action. This construction does not work in 3+1 dimensions since the Einstein-Hilbert action does not take the form of an action of a gauge theory in that case and for this reason we will be working in 2+1 dimensions throughout this section. In the usual picture of gauge theories Witten's construction can be understood as follows. In the absence of gravitational fields the underlying spacetime symmetry is the *global* Poincaré symmetry. If we want to construct a physical theory that is invariant under *local* Poincaré transformations we have to introduce new fields which compensate additional terms arising from this localization or gauging. It then turns out that these new fields represent the gravitational interaction.

The special role of the Poincaré group played in all this comes from the fact that it is the symmetry of spacetime in the absence of the gravitational field. However, we have seen in the previous section that when one considers the symmetry of spacetimes far away from any gravitational sources one in fact obtains an infinite dimensional generalization of the Poincaré group, the BMS group. Originally this result has been established at null infinity but recently it was shown at spatial infinity as well, see [5, 17] and as we have reviewed in section 3.3. This establishes BMS[8] as the symmetry of spacetime in the limit of a vanishing gravitational field.

Motivated by this observation we investigate in subsection 4.3 whether one can, in 2+1 dimensions, construct a gauge theory of gravity based on the algebra $\mathcal{B}^3$ defined in (2.37) instead of the Poincaré algebra. We are also going to discuss the asymptotic symmetry group of asymptotically Anti-de Sitter spacetimes, AdS_3 for short, as another potential gauge group.

The result, however, will be largely a negative one: the only gauge invariant action one can construct does not contain any gauge fields beyond the ones found in the standard Poincaré case.

[8]Our results of the precious section suggest that the asymptotic symmetry at spatial infinity might be even larger than BMS. However, since the BMS algebra is more established we will focus on it in this section.

4.2 Vielbein formalism

In this brief overview of the vielbein formalism we follow the explanations in [46]. A natural choice for the basis of the tangent space of a manifold at point p are the partial derivatives $\hat{e}_{(\mu)} = \partial_\mu$ with respect to the coordinates at that point. But same as well one can choose a different basis that is not related to a coordinate system and is given at each point of the manifold by basis vectors $\hat{e}_{(a)}$. We choose the set of basis vectors to be orthonormal so that for their inner product we have

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}, \tag{4.1}$$

where g is the metric tensor and η_{ab} the Minkowski metric. The old basis vectors can be expressed in terms of the new ones by

$$\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)}, \tag{4.2}$$

where the e^a_μ are referred to as vielbeins, or in four dimensions as tetrads and in three as triads. Using the vielbeins any tensor can be written in the new orthonormal basis, in particular we have for the metric tensor

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab} \tag{4.3}$$

and the vielbeins can therefore also be understood as a mapping to flat spacetime, which locally is always possible.

The covariant derivative of a tensor with components written in the vielbein basis is defined using the spin connection $\omega^a_{\mu b}$, for example

$$\nabla_\mu X^a = \partial_\mu X^a + \omega^a_{\mu b} X^b, \tag{4.4}$$

where the spin connection ω is related to the usual connection Γ as follows

$$\omega^a_{\mu b} = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda. \tag{4.5}$$

Curvature and torsion are defined in terms of the vielbein and spin connection as

$$R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^a_{\mu c} \omega^{cb}_\nu - \omega^a_{\nu c} \omega_\mu{}^{cb} \tag{4.6}$$

and

$$T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_\mu b^b_\nu - \omega^a_\nu b e^b_\mu. \tag{4.7}$$

Expressed in terms of the vielbein and spin connection one-forms

$$e^a = e^a_\mu dx^\mu, \quad \omega^{ab} = \omega^{ab}_\mu dx^\mu \tag{4.8}$$

they take the form

$$R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{bc}, \quad T^a = de^a + \omega^a_b \wedge e^b. \tag{4.9}$$

Notice that curvature and torsion take the form of a field strength tensor in gauge theories with e^a and ω^{ab} as gauge fields. The familiar expressions for Riemann and torsion tensor can be obtained by switching to greek indices

$$T^\rho_{\mu\nu} = e^\rho_a T^a_{\mu\nu}, \quad R^\rho_{\sigma\mu\nu} = e^\rho_a e_{\sigma b} R^{ab}_{\mu\nu}. \tag{4.10}$$

Finally, the Einstein-Hilbert action written in terms of vielbeins and the spin connection reads in four dimensions

$$I_{4d} = \frac{1}{2} \oint d^4x \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} e^a_\mu e^b_\nu R^{cd}_{\rho\sigma} \tag{4.11}$$

and in three dimensions

$$I_{3d} = \frac{1}{2} \oint d^3x \varepsilon^{\mu\nu\rho} \varepsilon_{abc} e^a_\mu R^{bc}_{\nu\rho}. \tag{4.12}$$

4.3 BMS gauge theory

4.3.1 Definition of gauge fields and transformation laws

In this subsection we investigate whether one can construct a gauge theory based on the BMS instead of the Poincaré group in 2+1 dimensions. We do so by generalizing the construction of Witten [6], which takes as a starting point the Poincaré algebra in the form

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = 0 \tag{4.13}$$

and expands the gauge fields in the generators of this algebra

$$A_i = e^a_i P_a + \omega^a_i J_a, \tag{4.14}$$

where we are writing indices μ, ν now as i, j to indicate that we are working in 2+1 dimensions. The vielbein e^a_i and spin connection $\omega^a_i = \epsilon^{abc}\omega_{ibc}$ are then interpreted as the gauge fields of the theory.

Instead of the Poincaré we take as a gauge group the BMS group defined by the commutation relations (2.37) and expand gauge fields A_i accordingly as

$$A_i = e^m_i t_m + \omega^m_i l_m, \tag{4.15}$$

with $m \in \mathbb{Z}$ and similarly expand the infinitesimal gauge parameter u

$$u = \rho^m T_m + \tau^m l_m. \tag{4.16}$$

The case when $m = 0, \pm 1$ corresponds to the standard Poincaré sector and we will occasionally refer to gauge fields outside of this sector as "higher gauge fields".

The variation of the gauge fields under a gauge transformation is defined as

$$\delta A_i = -\partial_i u - [A_i, u] \tag{4.17}$$

and upon evaluation we find the following transformation laws

$$\delta e_i^m = -\partial_i \rho^m + (2n - m) \left(e_i^{m-n} \tau^n + \omega_i^{m-n} \rho^n \right) \tag{4.18}$$

$$\delta \omega_i^m = -\partial_i \tau^m + (2n - m) \omega_i^{m-n} \tau^n. \tag{4.19}$$

The field strength tensor is defined as

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \tag{4.20}$$

and evaluating the commutator yields

$$\begin{aligned} F_{ij} &= S_{ij}^m T_m + R_{ij}^m l_m \\ &= \left(\partial_i e_j^m - \partial_j e_i^m + (m - 2n) \left(e_i^{m-n} \omega_j^n + \omega_i^{m-n} e_j^n \right) \right) T_m \\ &+ \left(\partial_i \omega_j^m - \partial_j \omega_i^m + (m - 2n) \omega_i^{m-n} \omega_j^n \right) l_m. \end{aligned} \tag{4.21}$$

Notice that S_{ij}^m and R_{ij}^m take the form of torsion and curvature defined in (4.6) and (4.7) but with generalized indices m, n. The curvature transforms under a gauge transformation as

$$\delta F_{ij} = -[F_{ij}, u] \tag{4.22}$$

which yields the following transformations laws

$$\delta S_{ij}^m = (2n - m) \left(S_{ij}^{m-n} \tau^n + R_{ij}^{m-n} \rho^n \right) \tag{4.23}$$

$$\delta R_{ij}^m = (2n - m) R_{ij}^{m-n} \tau^n. \tag{4.24}$$

4.3.2 Discussion of invariant form and construction of action

With these definitions at hand the next step is to construct a $\mathcal{B}^3$ gauge invariant Chern-Simons action, as before we closely follow the construction of [6]. The crucial ingredient for this construction is an element which is invariant in $\mathcal{B}^3$ (2.37), i.e. an element that commutes with all the generators also known as Casimir. In the case of the Poincaré algebra (4.13) such an element is given by $W = J^a P_a$. However, for $\mathcal{B}^3$ no such element is known to exist[9]. In a strict sense it is therefore not possible to construct a BMS gauge theory which would be on the same footing as the one based on the Poincaré group. Instead we construct the action using the invariant element from the Poincaré algebra which. As we will see in a moment, the resulting action generically contains higher gauge fields. But this action is not going to be invariant under all the transformations parametrized by (4.16). Our strategy will be to find the smallest set of gauge fields and transformation which we have to assume to vanish in order to restore gauge invariance.

The construction of the Chern-Simons action starts with the expression

$$I = \int F^m \wedge F^n d_{mn}, \tag{4.25}$$

where d_{mn} is an invariant quadratic form, in our case of the Poincaré group. The symbol F^m can stand for either S^m or R^m, depending on whether the l or T component is picked by d_{mn}. To determine d_{mn} one observes that the following combination of generators

$$W = l_m T_n d^{mn} = l_0 T_0 - \frac{1}{2}\left(l_{-1} T_1 + l_1 T_{-1}\right) \tag{4.26}$$

is invariant under Poincaré transformations and by inverting d^{mn} we find for d_{mn}

$$< l_0, T_0 >= 1, \quad < l_{-1}, T_1 >= -2, \quad < l_1, T_{-1} >= -2. \tag{4.27}$$

Evaluating (4.25) one finds for the action

$$\int_Y S^0 \wedge R^0 - 2(S^{-1} \wedge R^1 + S^1 \wedge R^{-1}), \tag{4.28}$$

where Y is a four manifold. To obtain from this the Chern-Simons action the integrand has to be written as a total derivative and upon using the divergence theorem the action reduces to an integral on a three manifold M

$$I = \int_M e^0 \wedge R^0 - 2\left(e^{-1} \wedge R^1 + e^1 \wedge R^{-1}\right). \tag{4.29}$$

From the definition of the curvature (4.21) one can see that $R^{0,\pm 1}$ contains gauge fields outside of the Poincaré sector. By construction this action is invariant under the Poincaré group if all higher gauge fields are vanishing but it could potentially exhibit a larger symmetry.

To check whether this is the case we calculate the variation of the action and applying

[9] As stated in [47] no classification of the Casimirs of $\mathcal{B}^3$ is known. However, therein an argument is made that the only Casimirs of $\mathcal{B}^3$ are so-called central charges. These are not constructed from the generators of the algebra, they instead arise from a certain extension of the algebra, and are therefore not of interest for us.

the transformation laws (4.18) and (4.24) we obtain

$$\delta I = 2\int d^3x \epsilon^{ijk}\Big[\frac{1}{2}\left(2ne_i^{-n}\tau^n + 2n\omega_i^{-n}\rho^n - \partial_i\rho^0\right)R^0_{jk} + ne_i^0 R^{-n}_{jk}\tau^n$$
$$-\left((1+2n)e_i^{-1-n}\tau^n + (1+2n)\omega_i^{-1-n}\rho^n - \partial_i\rho^{-1}\right)R^1_{jk} + (1-2n)e_i^{-1}R^{1-n}_{jk}\tau^n$$
$$+\left((1-2n)e_i^{1-n}\tau^n + (1-2n)\omega_i^{1-n}\rho^n + \partial_i\rho^1\right)R^{-1}_{jk} - (1+2n)e_i^1 R^{-1-n}_{jk}\tau^n\Big], \quad (4.30)$$

where each line is of the form $\delta e \wedge R + e \wedge \delta R$ and all $n \in \mathbb{Z}$. For the action to be invariant its variation has to vanish.

Even though the construction of the action guarantees Poincaré gauge invariance if higher gauge fields are vanishing, this invariance is not immediately obvious. In a first step we therefore check that the variation of the action indeed vanishes when we restrict gauge fields and transformations to the Poincaré case, i.e. when we assume that $e_i^n, \omega_i^n, \rho^n, \tau^n$ and R^n_{ij} vanish for $n \neq \pm 1, 0$. In a second step we will consider the general case by allowing all transformations and gauge fields to be finite.

One can directly see that in the Poincaré case the terms containing e_i^n cancel in (4.30). In order to see that the remaining terms cancel as well we plug in the definition of R^n_{jk} from (4.21) and for terms not containing any derivatives we find

$$-\epsilon^{ijk}\rho^n\omega_k^n\left[4n^2\omega_i^{-n}\omega_j^{-n} + (1-4n^2)\left(\omega_i^{-1-n}\omega_j^{1-n} + \omega_i^{1-n}\omega_j^{-1-n}\right)\right], \quad (4.31)$$

and since the first term and the term in brackets are symmetric in i, j and are contracted with the fully antisymmetric tensor they both vanish. For the terms containing derivatives we obtain after performing several integrations by parts and rearrangement of indices the remaining terms, up to boundary terms

$$2\epsilon^{ijk}\Big[\rho^n\left(-(1+2n)\omega_i^{-1-n}\partial_j\omega_k^1 + (1-2n)\omega_i^{1-n}\partial_j\omega_k^{-1} + n\omega_i^{-n}\partial_j\omega_k^0\right)$$
$$- n\rho^0\omega_i^n\partial_j\omega_k^{-n} + (1+2n)\rho^1\omega_i^n\partial_j\omega_k^{-1-n} - (1-2n)\rho^{-1}\omega_i^n\partial_j\omega_k^{1-n}\Big], \quad (4.32)$$

which also cancel. Thus the action (4.29) is indeed Poincaré gauge invariant and we proceed now with the general case.

If we assume all fields and transformations to be finite there are several non-vanishing terms in the variation of the action, which means that the action build from the most general gauge fields is not invariant under general $\mathcal{B}^3$ transformations. We can, however, introduce constraints, i.e. demands that certain fields and transformations are vanishing, that let the variation vanish for an action that contains as many higher gauge fields as possible and is invariant under transformations which are as general as possible. Consider in (4.30) the term

$$ne_i^0 R^{-n}_{jk}\tau^n \quad (4.33)$$

which can not cancel(except in the Poincaré case, of course) with any of the other terms and we thus have to constrain either $R^x_{jk} = 0$ or $\tau^x = 0$, where x denotes all n except for $n = \pm 1, 0$. Similarly from the term

$$ne_i^{-n}\tau^n R^0_{jk} \tag{4.34}$$

we find the constraint $e^x_i = 0$ or $\tau^x = 0$. Another constraint comes from terms like

$$(1+2n)e_i^{-1-n}\tau^n, \tag{4.35}$$

since $e^x_i = 0$ still allows a term $-3e^1_i\tau^{-2}$. The terms proportional to $\tau^{\pm 2}$ do not cancel and we therefore need to impose $\tau^{\pm 2} = 0$. The terms in (4.31) vanish without imposing further constraints. From the term in (4.32)

$$-n\rho^0\omega^n_i\partial_j\omega^{-n}_k \tag{4.36}$$

we have $\omega^x_i = 0$, since it can not cancel with any other term. This constraint makes the remaining terms in (4.32) vanish as well. To see this consider the term

$$-(1+2n)\epsilon^{ijk}\rho^n\omega_i^{-1-n}\partial_j\omega^1_k, \tag{4.37}$$

where $\omega^x_i = 0$ allows a term $\epsilon^{ijk}3\rho^{-2}\omega^1_i\partial_j\omega^1_k$, which vanishes because $\omega^1_i\partial_j\omega^1_k$ is symmetric in i, k. The other terms vanish similarly without having to constrain ρ. From the condition $\omega^x_i = 0$ it follows that $R^x_{ij} = 0$ as can be seen from the definition (4.21).

To summarize, there are two different choices of constraints to restore invariance of the action (4.29)

$$R^n_{ij} = e^n_i = \omega^n_i = \tau^{\pm 2} = 0 \quad \text{or} \quad R^n_{ij} = e^{\pm 2}_i = \omega^n_i = \tau^n = 0 \quad \text{for } n \neq \pm 1, 0 \tag{4.38}$$

and the vanishing of higher ω^n_i directly implies that there are no extra fields present in the action compared to the standard Poincaré case. Since we do not need to constrain ρ^n, the action (4.29) is, assuming above conditions, invariant under general supertranslations. This invariance is however trivial in the sense that under the condition $\omega^x_i = 0$ the fields which are present in the action are invariant themselves under supertranslations, as can be seen from the transformations (4.18) and (4.19). There is one exception from this, namely the change generated in $e^{\pm 1}$ by $\rho^{\pm 2}$ which reads $\delta e^{\pm 1}_i = (2n \mp 1)\omega^{\mp 1}\rho^{\pm 2}$. In the first choice above the action is also invariant under almost all superrotations, but again this invariance is trivial since the fields in the action are themselves invariant under these superrotations due to the conditions $e^x_i = \omega^x_i = 0$.

Our final result is that the gauge symmetry of the action (4.29) is the Poincaré group and additionally supertranslations generated by $\rho^{\pm 2}$. This symmetry is present only in the case that the action contains no higher gauge fields.

We note that instead of the invariant element (4.26) we could have chosen any of the invariant elements corresponding to the embeddings in (2.42), (2.43). The discussion we just presented would go through exactly the same, with ± 1 appropriately replaced by $\pm n$. Choosing any of these different invariant forms does therefore not lead to a qualitatively different result but amounts to a mere renaming of indices.

4.3.3 Construction of gauge invariant action with finite cosmological constant

Now we consider the case of a finite, negative cosmological constant Λ. The setup and following discussion is very similar to the previous subsection but instead of $\mathcal{B}^3$ we take as a starting point the following algebra, see [48, 49]

$$\begin{aligned} [l_m, l_n] &= (m-n) l_{m+n}, \\ [T_m, T_n] &= \Lambda (m-n) l_{m+n}, \\ [l_m, T_n] &= (m-n) T_{m+n}, \end{aligned} \tag{4.39}$$

which we will refer to as $\Lambda\mathcal{B}^3$ and which describes the asymptotic symmetry of spacetimes which asymptotically approach three dimensional Anti-de Sitter spacetime, AdS_3. Similarly to the asymptotically flat case we obtain from (4.39) the symmetry algebra of isometries of exactly AdS_3 spacetime by constricting m, n to values $0, \pm 1$. We restrict ourselves to a negative cosmological constant since in that case spacelike hypersurfaces are open and there exists a non-trivial asymptotic symmetry at spatial infinity. This asymptotic symmetry has to be included in the Hamiltonian of the theory in the form of surface integrals, as was explained for the case $\Lambda = 0$ in chapter 3. For asymptotic de Sitter spaces such hypersurfaces are compact and there is no asymptotic structure [50].

We proceed in analogy to the $\Lambda = 0$ case. Evaluating (4.17) with (4.39) yields the following transformation laws for the gauge fields

$$\delta e_i^m = -\partial_i \rho^m + (2n - m) \left(e_i^{m-n} \tau^n + \omega_i^{m-n} \rho^n \right) \tag{4.40}$$

$$\delta \omega_i^m = -\partial_i \tau^m + (2n - m) \left(\omega_i^{m-n} \tau^n + \Lambda e_i^{m-n} \rho^n \right) \tag{4.41}$$

and from (4.20) it follows

$$S_{ij}^m = \partial_i e_j^m - \partial_j e_i^m + (m - 2n) \left(e_i^{m-n} \omega_j^n + \omega_i^{m-n} e_j^n \right) \tag{4.42}$$

$$R_{ij}^m = \partial_i \omega_j^m - \partial_j \omega_i^m + (m - 2n) \left(\omega_i^{m-n} \omega_j^n + \Lambda e_i^{m-n} e_j^n \right). \tag{4.43}$$

As in the previous subsection the algebra (4.39) has no invariant element and we choose an invariant element of the AdS_3 symmetry algebra to construct the Chern-Simons action. The element (4.26) is invariant under (4.39) as well, but with the cosmological constant non-vanishing there is a second non-degenerate invariant form, namely

$$< t_0, t_0 >= \Lambda, \quad < t_1, t_{-1} >=< t_{-1}, t_1 >= -2\Lambda \tag{4.44}$$

$$< l_0, l_0 >= 1, \quad < l_1, l_{-1} >=< l_{-1}, l_1 >= -2 \tag{4.45}$$

and we are choosing now this one to construct the action. From (4.25) we obtain for the action

$$\begin{aligned} I = \int d^4x \epsilon^{ijkl} \Big[&\Lambda \left(S^0_{ij} S^0_{kl} - 2(S^1_{ij} S^{-1}_{kl} + S^{-1}_{ij} S^1_{kl}) \right) \\ &+ R^0_{ij} R^0_{kl} - 2(R^1_{ij} R^{-1}_{kl} + R^{-1}_{ij} R^1_{kl}) \Big]. \end{aligned} \tag{4.46}$$

As before, the integrand can be written as a total derivative and using the divergence theorem we find the Chern-Simons action

$$\begin{aligned} I = 2 \int d^3x \epsilon^{ijk} \Big[&f_{mn} \omega^m_i (\partial_j \omega^n_k - \partial_k \omega^n_j) - 4(1-2n) \omega^{-1}_i \omega^{1-n}_j \omega^n_k \\ &+ \Lambda f_{mn} e^m_i (\partial_j e^n_k - \partial_k e^n_j) + 4\Lambda \Big(- n \omega^0_i e^{-n}_j e^n_k \\ &+ (1+2n) \omega^1_i e^{-1-n}_j e^n_k - (1-2n) \omega^{-1}_i e^{1-n}_j e^n_k \Big) \Big]. \end{aligned} \tag{4.47}$$

Under a transformation of the gauge fields the action transforms as

$$\begin{aligned} \delta I = 8 \int d^3x \epsilon^{ijk} \big[&f_{mn} \delta\omega^m_i \partial_j \omega^n_k - (1-2n) \left(2\omega^{-1}_i \delta\omega^{1-n}_j \omega^n_k + \delta\omega^{-1}_i \omega^{1-n}_j \omega^n_k \right) \\ &+ \Lambda \big(f_{mn} \delta e^m_i \partial_j e^n_k - n \left(2\omega^0_i \delta e^{-n}_j e^n_k + \delta\omega^0_i e^{-n}_j e^n_k \right) \\ &+ (1+2n) \left(2\omega^1_i \delta e^{-1-n}_j e^n_k + \delta\omega^1_i e^{-1-n}_j e^n_k \right) \\ &- (1-2n) \left(2\omega^{-1}_i \delta e^{1-n}_j e^n_k + \delta\omega^{-1}_i e^{1-n}_j e^n_k \right) \big) \big], \end{aligned} \tag{4.48}$$

up to boundary terms since integration by parts was used to combine terms explicitly containing derivatives. First, we check that the action is indeed invariant under the AdS_3 isometry transformations, i.e. we assume that $e^n_i, \omega^n_i, \rho^n, \tau^n$ and R^n_{ij} vanish for $n \neq \pm 1, 0$. Plugging in the transformation laws (4.40) and (4.41) we find the following terms not containing Λ nor partial derivatives

$$\begin{aligned} &- (1-2n) \omega^n_k \tau^l \epsilon^{ijk} \left(2(2l+n-1) \omega^{-1}_i \omega^{1-n-l}_j + (2l+1) \omega^{-1-l}_i \omega^{1-n}_j \right) \\ &= \tau^l \epsilon^{ijk} \left(-2(2l-1) \omega^{-1}_i \omega^{1-l}_j \omega^0_k + 2(2l+1) \omega^{-1-l}_i \omega^0_j \omega^1_k + 4l \omega^{-1}_i \omega^{-l}_j \omega^{-1}_k \right) \end{aligned} \tag{4.49}$$

and terms not containing Λ but partial derivatives

$$\begin{aligned} \epsilon^{ijk} \big[&\tau^n \left(-2n \partial_j \omega^0_i \omega^{-n}_k + 2(2n+1) \partial_j \omega^1_i \omega^{-1-n}_k + 2(2n-1) \partial_j \omega^{-1}_i \omega^{1-n}_k \right) \\ &+ (1-2n) \left(-2\partial_j (\omega^{-1}_i \omega^n_k) \tau^{1-n} + \partial_j (\omega^{1-n}_i \omega^n_k) \tau^{-1} \right) \big] \end{aligned} \tag{4.50}$$

and upon carrying out the sums all terms either cancel or vanish because they are of the form $\epsilon^{ijk}\omega_i^n\omega_j^n$. Next, we consider terms proportional to Λ which do not contain partial derivatives

$$\begin{aligned}\Lambda\epsilon^{ijk}\big[&-(1-2n)\omega_k^n\rho^l\left(2(2l-1+n)\omega_i^{-1}e_j^{1-n-l}+(2l+1)e_i^{-1-l}\omega_j^{1-n}\right)\\&-ne_k^n\left(2(2l+n)\omega_i^0\left(e_j^{-n-l}\tau^l+\omega_j^{-n-l}\rho^l\right)+2l\omega_i^{-l}\tau^le_j^{-n}\right)\\&+(1+2n)e_k^n\Big(2(2l+1+n)\omega_i^1(e_j^{-1-n-l}\tau^l+\omega_j^{-1-n-l}\rho^l)\\&\quad+(2l-1)\omega_i^{1-l}\tau^le_j^{-1-n}\Big)\\&-(1-2n)e_k^n\Big(2(2l-1+n)\omega_i^{-1}\left(e_j^{1-n-l}\tau^l+\omega_j^{1-n-l}\rho^l\right)\\&\quad+(2l+1)\omega_i^{-1-l}\tau^le_j^{1-n}\Big)\big]\end{aligned}\tag{4.51}$$

and summing over n and collecting terms gives

$$\begin{aligned}\Lambda\epsilon^{ijk}\Big(&\rho^l\big[2(2l+1)e_i^{-1-l}\omega_j^0\omega_k^1+2(2l-1)e_i^{1-l}\omega_j^{-1}\omega_k^0-4le_i^{-l}\omega_j^{-1}\omega_k^1\\&\quad+2e_i^0\left((2l-1)\omega_j^{1-l}\omega_k^{-1}-(2l+1)\omega_j^{-1-l}\omega_k^1\right)\\&\quad+2e_i^1\left((2l+1)\omega_j^{-1-l}\omega_k^0-2l\omega_j^{-l}\omega_k^{-1}\right)\\&\quad-2e_i^{-1}\left((2l-1)\omega_j^{1-l}\omega_k^0-2l\omega_j^{-l}\omega_k^1\right)\big]\\&+\tau^l\big[2(2l-1)\omega_i^{1-l}e_j^{-1}e_k^0+2(2l+1)\omega_i^{-1-l}e_j^0e_k^1+4l\omega_i^{-l}e_j^1e_k^{-1}\\&\quad+2\omega_i^0\left((2l-1)e_j^{1-l}e_k^{-1}-(2l+1)e_j^{-1-l}e_k^1\right)\\&\quad+2\omega_i^1\left((2l+1)e_j^{-1-l}e_k^0-2le_j^{-l}e_k^{-1}\right)\\&\quad-2\omega_i^{-1}\left((2l-1)e_j^{1-l}e_k^0-2le_j^{-l}e_k^1\right)\big]\Big)\end{aligned}\tag{4.52}$$

and one can check that for each l all the terms cancel out. Next come terms which contain both Λ and partial derivatives

$$\begin{aligned}\Lambda\epsilon^{ijk}\big[&f_{mn}(2l-m)\left(e_i^{m-l}\rho^l\partial_j\omega_k^n+e_i^{m-l}\tau^l\partial_je_k^n+\omega_i^{m-l}\rho^l\partial_je_k^n\right)\\&+n\left(2\omega_i^0\partial_j\rho^{-n}+\partial_i\tau^0e_j^{-n}\right)e_k^n\\&-(1+2n)\left(2\omega_i^1\partial_j\rho^{-1-n}+\partial_i\tau^1e_j^{-1-n}\right)e_k^n\\&+(1-2n)\left(2\omega_i^{-1}\partial_j\rho^{1-n}+\partial_i\tau^{-1}e_j^{1-n}\right)e_k^n\big]\end{aligned}\tag{4.53}$$

and performing integration by parts on the three last lines and summing over f_{mn} yields up to boundary terms

$$\begin{aligned}
&\Lambda\epsilon^{ijk}\Big(\rho^l\big[2l\omega_i^{-l}\partial_j e_k^0 - 2(2l-1)\omega_i^{1-l}\partial_j e_k^{-1} - 2(2l+1)\omega_i^{-1-l}\partial_j e_k^1 \\
&+ 2l\omega_i^0\partial_j e_k^{-l} - 2(2l+1)\omega_i^1\partial_j e_k^{-1-l} + 2(1-2l)\omega_i^{-1}\partial_j e_k^{1-l}\big] \\
&+ \tau^l\big[-2le_j^{-l}\partial_i e_k^0 + 2(2l-1)e_j^{1-l}\partial_i e_k^{-1} + 2(2l+1)e_j^{-1-l}\partial_i e_k^1\big] \\
&- 2l\tau^0 e_j^{-n}\partial_i e_k^l + 2(2l+1)\tau^1 e_j^{-1-l}\partial_i e_k^l - (1-2l)\tau^{-1}e_j^{1-n}\partial_i e_k^l\Big),
\end{aligned} \tag{4.54}$$

where any terms proportional to $\partial_i\omega_j^m$ have already canceled out and again for each l the remaining terms can be seen to cancel. Finally, there are terms proportional to Λ^2

$$\begin{aligned}
&\Lambda^2 e_k^n\rho^l\epsilon^{ijk}\Big(-2nle_i^{-l}e_j^{-n} + (1+2n)(2l-1)e_i^{1-l}e_j^{-1-n} \\
&\qquad - (1-2n)(2l+1)e_i^{-1-l}e_j^{1-n}\Big) \\
&= \Lambda^2\rho^l\epsilon^{ijk}\left(4le_i^l e_j^1 e_k^{-1} + 2(2l-1)e_i^{1-l}e_j^{-1}e_k^0 + 2(2l+1)e_i^{-1-l}e_j^0 e_k^1\right)
\end{aligned} \tag{4.55}$$

and the terms either cancel each other or they vanish since they are for the form $\epsilon^{ijk}e_i^n e_j^n e_k^m$. We have thus shown that the action (4.47) is indeed gauge invariant under the isometries of AdS_3 if all higher fields and transformations are assumed to vanish and we now proceed with the general case.

If we allowed for all gauge fields and transformations to be non-vanishing many extra terms in the variation would appear of the action which do not cancel out and as before we will try to find the smallest set of gauge fields and transformation we have to assume to vanish in order to restore gauge invariance. To let the extra terms vanish in (4.49) we impose $\omega_i^x = 0$, which still allows terms proportional to $\omega_i^{\mp1}\omega_j^{\mp1}\tau^{\pm2}$, but since they are symmetric and vanishing we do not have to constrain τ^n. With the same reasoning the constraint $\omega_i^x = 0$ is also sufficient to make the extra terms in (4.50) vanish. From the terms in (4.51) we find that we need to restrict $e_i^x = \tau^{\pm2} = \rho^{\pm2} = 0$, this leaves terms proportional to $\rho^{\pm3}, \tau^{\pm3}$ but they vanish again because they are symmetric. These constraints are sufficient to also make all the extra terms in (4.53) and (4.55) vanish.

To summarize, in order to restore gauge invariance of the action (4.47) we need to impose the following restrictions

$$e_i^n = \omega_i^n = \tau^{\pm2} = \rho^{\pm2} = 0, \quad \text{for } n \neq \pm1, 0. \tag{4.56}$$

This means that there are no extra fields in the action and that the action is invariant under $\Lambda\mathcal{B}^3$ gauge transformations, except for the cases $\tau^{\pm2}$ and $\rho^{\pm2}$. However, from the transformation laws of the gauge fields (4.40) and (4.41) we find that the invariance of the action under these $\Lambda\mathcal{B}^3$ gauge transformations is trivial in the sense that the gauge fields themselves are invariant under these transformations.

5 κ-deformed BMS symmetry

5.1 Introduction

In this section we perform a so-called κ-deformation [7]–[11] of the BMS symmetry and investigate the properties of the deformed algebra. The term deformation here refers to a generalization of an algebra that reduces to the undeformed algebra in an appropriate limit of the deformation parameter, in the case considered here this limit is $\kappa \to \infty$.

The main motivation to study κ-deformations is that it is thought [18] that the κ-deformed Poincaré symmetry is related to quantum gravity, which remains elusive even after decades of research. It is believed to be so since the deformation parameter κ has the dimension of mass by construction and can therefore naturally be identified with the quantum gravity energy scale, the Planck mass, $M_{Pl} \sim 10^{19}$ GeV. Another widely held belief is that the structure of spacetime drastically changes at the Planck scale, $l \sim 10^{-35}$ m, [51] and that the symmetries of such quantum spacetimes differ from the classical Poincaré symmetries as well. One could expect that even in the limit of vanishing spacetime curvature the structure of spacetime at this scale remains different from the classical case and it is thought that κ-Poincaré describes the symmetry of such a "flat quantum spacetime". Such a link between a proposed theory of quantum gravity and κ-Poincaré has been established in the case of three-dimensional gravity [20] but not in the four-dimensional case.

One way of thinking about the BMS symmetry is that GR does not reduce to special relativity at large distances and weak fields but instead there remains a large space of inequivalent vacua, as has been explained in section 2.1. One could therefore argue that the symmetries of the aforementioned flat quantum spacetime is in fact described by a deformed BMS rather than a deformed Poincaré symmetry. This observation motivates us to extent the κ-deformation of the Poincaré algebra to the infinite-dimensional BMS algebra. But before performing the actual algebra deformation we are giving a short review of the necessary mathematical notions such as Hopf algebra, Lie bialgebra and Drinfeld's twist deformation [52, 53].

5.2 Review of mathematical notions

It is in fact not the Poincaré or BMS algebra, which are Lie algebras, that are being deformed directly but the associated Hopf algebras, which can be thought of as algebras equipped with additional structures. We are first going to define what exactly a Hopf algebra is and then we will show how a Hopf algebra can be obtained from a given Lie algebra.

To define a Hopf algebra we follow the explanations in [54]. We begin with the definition of an associative, unital algebra which is a vector space V over a field K, e.g. real or complex numbers, together with a multiplication(or product) $m : V \otimes V \to V$ and unit

$\eta : K \to A$ such that

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \tag{5.1}$$

$$m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes \eta). \tag{5.2}$$

The product, as it is usually written, is given by $ab := m(a \otimes b)$ and the first line then simply expresses associativity $a(bc) = (ab)c$. The unit is determined by its value $\eta(1) \in V$ and the second line can then be recognized as the definition of the unit $a1 = 1a = a$.

Next, we define a coalgebra which is a vector space V over a field K together with a coproduct $\Delta : V \to V \otimes V$ and counit ε such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{5.3}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta. \tag{5.4}$$

By comparison with the previous definition we can see that the coproduct "dualizes" the notion of the product: instead of mapping from two elements of V to one element it does the reverse, it maps one element of V to two elements. Later in subsection 5.5 we show in what context this construction is used which will also make its meaning more clear. For now we are just interested in the mathematical structure.

Finally, a Hopf algebra is both an algebra and a coalgebra obeying certain compatibility conditions, namely

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(1) = 1. \tag{5.5}$$

Additionally it contains the antipode $S : V \to V$ which fulfills

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta, \quad S(ab) = S(b)S(a) \tag{5.6}$$

and, as will be made more clear in 5.5, can be understood as a generalization of the inverse.

With this definition of a Hopf algebra H at hand we now explain how one can obtain one from a given Lie algebra g. The first step is to construct the universal enveloping algebra $U(g)$. This construction is necessary since H is defined to be a unital, associative algebra. Intuitively one can think of $U(g)$ as g being embedded into an unital, associative algebra A such that the abstract bracket $[x, y]$ in g is realized by the commutator $xy - yx$ in A. Formally, $U(g)$ is defined as the quotient space $U(g) = T(g)/J(g)$, where $T(g)$ is the tensor algebra of g, i.e. the the direct sum of all possible tensor products of vectors in g, and $J(g) = x \otimes y - y \otimes x = [x, y]$.

As an example for an universal enveloping algebra we consider the Lie algebra sl(2,$\mathbb{C}$) which is spanned by three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy $[H, X] = 2X, [H, Y] = -2Y$ and $[X, Y] = H$. The universal enveloping algebra is then spanned by all (non-negative) powers of three elements x, y, h, therefore also contains the unit, and is associative. The only additional conditions the enveloping algebra is subject to are

$$hx - xh = 2x, \quad hy - yh = -2y, \quad xy - yx = h$$

and notice that therefore it is not true that, for example, $x^2 = 0$ even though from the above matrix representation it follows that $X^2 = 0$.

The enveloping algebra $U(g)$ can then be equipped with a coproduct Δ_0, counit ε_0 and antipode S_0 to form a so-called primitive Hopf algebra H. These structures take the following form for all $X \in g$ [54]

$$\begin{aligned} \Delta_0(X) = X \otimes 1 + 1 \otimes X, \quad S_0(X) = -X, \quad \varepsilon_0(X) = 0, \quad \varepsilon_0(1) = 1, \\ S_0(1) = 1, \quad \Delta_0(1) = 1 \otimes 1. \end{aligned} \tag{5.7}$$

One can check that the undeformed structures (5.7) are indeed consistent with the previous definition of Hopf algebra, for instance one finds

$$m \circ (S \otimes \mathrm{id})(1 \otimes X + X \otimes 1) = 0 = \varepsilon(X)1. \tag{5.8}$$

Such standard Hopf algebra structure will be then deformed by using Drinfeld's twist deformation techniques. To this end one has to extend $U(g)$ by introducing a new commuting generator denoted as $1/\kappa$. A new, deformed, Hopf algebra equipped with a twist-deformed coproduct Δ and a compatible antipode S becomes, according to Drinfeld's terminology, a 'quantum group'. The coproduct in our case will be obtained by a similarity transformation

$$\Delta(X) = \mathcal{F} \Delta_0(X) \mathcal{F}^{-1} \tag{5.9}$$

with a twist $\mathcal{F} \equiv a_i \otimes b_i \in H \otimes H$ that has to satisfy the 2-cocycle and normalization conditions

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)(\mathcal{F}), \quad \varepsilon(a_i) b_i = 1, \tag{5.10}$$

where $\mathcal{F}_{12} = a_i \otimes b_i \otimes 1$ etc. and repeated indices indicate a summation. The deformed antipode is given by

$$S = v S_0 v^{-1}, \tag{5.11}$$

where $v = m \circ (\mathrm{id} \otimes S)(\mathcal{F})$ and v^{-1} is its inverse. The twist deformation only modifies the coproduct and antipode but leaves the algebra g unchanged. This is in contrast to the time-like κ-Poincaré, see e.g. [11], where the algebra sector is deformed as well.

The 2-cocycle condition guarantees that the deformed coproduct remains coassociative which can be seen as follows:

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta(X) &= (\mathcal{F}\Delta_0\mathcal{F}^{-1} \otimes \mathrm{id})(\mathcal{F}\Delta_0(X)\mathcal{F}^{-1}) \\ &= (\mathcal{F} \otimes \mathrm{id})(\Delta_0 \otimes \mathrm{id})(\mathcal{F}\Delta_0(X)\mathcal{F}^{-1})(\mathcal{F}^{-1} \otimes \mathrm{id}) \\ &= [\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})(\mathcal{F})][(\Delta_0 \otimes \mathrm{id})\Delta_0(X)][(\Delta_0 \otimes \mathrm{id})(\mathcal{F}^{-1})\mathcal{F}_{12}^{-1}] \\ &= [\mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)(\mathcal{F})][(1 \otimes \Delta_0)(\Delta_0(X))][(\mathrm{id} \otimes \Delta_0)(\mathcal{F}^{-1})\mathcal{F}_{23}^{-1}] \\ &= \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)(\mathcal{F})(\mathcal{F}\Delta_0\mathcal{F}^{-1})F_{23}^{-1} \\ &= (\mathrm{id} \otimes \Delta) \circ \Delta(X), \end{aligned} \tag{5.12}$$

where in the second line we used that $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, in the third line that Δ_0 is a homomorphism and in the fourth line we used in the first and last bracket the 2-cocycle condition and in the middle bracket coassociativity of Δ_0. Notice that in the fourth line in the last bracket we actually used the inversed form of the 2-cocycle condition:

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0) \tag{5.13}$$

$$\Rightarrow (\Delta_0 \otimes \mathrm{id})(\mathcal{F}^{-1})\mathcal{F}_{12}^{-1} = (\mathrm{id} \otimes \Delta_0)(\mathcal{F}^{-1})\mathcal{F}_{23}^{-1}. \tag{5.14}$$

For every Hopf algebra which is obtained as a quantum deformation of a Lie algebra (so-called quantized universal enveloping algebra) one introduces a formal variable enabling one to expand the deformed coproduct and antipode. In our case this is the dimension full parameter $1/\kappa$. It turns out that for each coproduct the first order contribution of the anti-symmetrized coproduct defines the so-called "classical limit". Applied to the Lie algebra generators $X \in g$

$$\delta(X) = \lim_{\kappa \to \infty} \kappa(\Delta(X) - \Delta^{21}(X)) \tag{5.15}$$

this defines on the initial Lie algebra g the cobracket δ providing the structure of a Lie bialgebra, where Δ^{21} denotes the coproduct with flipped legs, i.e. if $\Delta(X) = X_{1a} \otimes X_{2a}$ then $\Delta^{21}(X) = X_{2a} \otimes X_{1a}$. As will be explained in more detail below δ is compatible with the bracket in g and satisfies a dual Jacobi identity. Also notice that in the undeformed case δ is vanishing since Δ_0 is symmetric. Let us briefly discuss some properties of such Lie bialgebras following the treatment in [55, 56].

First we need to define a Lie coalgebra which is a vector space equipped with a cobracket $\delta : g \to g \otimes g$ that is anti-symmetric and fulfills the dual Jacobi identity, for all $X \in g$

$$\left(\mathrm{id} + \sigma + \sigma^2 \right) (\delta \otimes \mathrm{id})\delta(X) = 0, \tag{5.16}$$

where σ denotes a cyclic permutation of $g \otimes g \otimes g$. A Lie bialgebra is then defined as a set of a Lie algebra and Lie coalgebra that are compatible in the following sense

$$\begin{aligned}\delta([X,Y]) &= \mathrm{ad}_X\delta(Y) - \mathrm{ad}_Y\delta(X) \\ &= (\mathrm{ad}_X \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_X)\delta(Y) - (\mathrm{ad}_Y \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_Y)\delta(X).\end{aligned} \tag{5.17}$$

If the cobracket is implemented by an antisymmetric element $r \in g \wedge g$, the so-called classical r-matrix, satisfying the classical Yang-Baxter equation (see below) the Lie bialgebra is called coboundary. In that case we obtain a cobracket applied to an arbitrary element X of the initial Lie algebra in the form

$$\delta_r(X) = \mathrm{ad}_X r \equiv [X \otimes \mathrm{id} + \mathrm{id} \otimes X, r]. \tag{5.18}$$

This form of the cobracket automatically satisfies the following compatibility condition (5.17). To check that this is true we first note that we can write

$$\delta_r(X) = (\mathrm{ad}_X \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_X)(r) \tag{5.19}$$

and substituting in (5.17) we have

$$\begin{aligned}\delta_r([X,Y]) &= (\mathrm{ad}_X \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_X)(\mathrm{ad}_Y \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_Y)(r) \\ &\quad - (\mathrm{ad}_Y \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_Y)(\mathrm{ad}_X \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_X)(r) \\ &= \big((\mathrm{ad}_X\mathrm{ad}_Y - \mathrm{ad}_Y\mathrm{ad}_X) \otimes \mathrm{id} + \mathrm{id} \otimes (\mathrm{ad}_X\mathrm{ad}_Y - \mathrm{ad}_Y\mathrm{ad}_X)\big)(r) \\ &= (\mathrm{ad}_{[X,Y]} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_{[X,Y]})(r) = \mathrm{ad}_{[X,Y]}(r),\end{aligned} \tag{5.20}$$

which shows that indeed (5.17) follows from (5.18). In the last line we used that for $Z \in g$

$$\mathrm{ad}_X\mathrm{ad}_Y Z - \mathrm{ad}_Y\mathrm{ad}_X Z = \mathrm{ad}_{[X,Y]}Z \tag{5.21}$$

which we can also write as

$$[X,[Y,Z]] - [Y,[X,Z]] = [[X,Y],Z] \tag{5.22}$$

and this is equivalent to the Jacoby identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{5.23}$$

For coboundary deformations the flipped coproduct is provided by the quantum r-matrix $\mathcal{R}$: $\Delta^{21} = \mathcal{R}\Delta\mathcal{R}^{-1}$. In this way the formula (5.15) expresses on the one hand the so-called classical limit of quantum deformations and on the other hand relations between classical and quantum r-matrices. Furthermore, the quantum r-matrix is related to the twist via $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ and thus (5.15) also expresses a relation between the twist and the classical r-matrix.

As stated above the r-matrix has to fulfill the classical Yang-Baxter equation

$$[[r, r]] = \Omega, \tag{5.24}$$

where $[[,]]$ denotes the Schouten bracket defined as

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \tag{5.25}$$

and Ω is an adjoint invariant element, i.e. $\mathrm{ad}_X \Omega = 0$.

Note that sometimes (5.24) is referred to as modified Yang-Baxter equation and only the case $\Omega = 0$ is called classical. If the rhs of (5.24) is zero the corresponding bialgebra structure is called coboundary triangular. All such deformations are derivable by a twisting procedure (like eq.(5.9)) as described in [36, 57] and references therein.

The classical r-matrix is required to fulfill (5.24) because this condition guarantees that δ satisfies the dual Jacobi identity (5.16) which follows from the following identity

$$(\mathrm{id} + \sigma + \sigma^2)(\delta \otimes \mathrm{id})\delta(X) = \mathrm{ad}_X[[r, r]], \tag{5.26}$$

where σ denotes a cyclic permutation of the element $g \otimes g \otimes g$. The proof of this identity is quite lengthy and can be found in [56] and therefore, instead of presenting this proof, we demonstrate that it holds true for a simple example. To this end we consider the algebra with just two elements $[x, y] = x$ and r-matrix given by $r = x \wedge y$ so that from (5.18) we obtain

$$\delta(x) = 0, \quad \delta(y) = -x \wedge y. \tag{5.27}$$

Evaluating each term in the Schouten bracket we find

$$\begin{aligned}
&[r_{12}, r_{13}] = -[y, x] \otimes x \otimes y - [x, y] \otimes y \otimes x = x \otimes x \otimes y - x \otimes y \otimes x \\
&[r_{12}, r_{13}] = -x \otimes x \otimes y + y \otimes x \otimes x \\
&[r_{13}, r_{23}] = x \otimes y \otimes x - y \otimes x \otimes x
\end{aligned} \tag{5.28}$$

and since the sum of these terms vanishes the Lie bialgebra in this example is triangular. Now we evaluate the lhs of (5.26)

$$\begin{aligned}
(\delta \otimes \mathrm{id})\delta(x) &= 0 \\
(\delta \otimes \mathrm{id})\delta(y) &= -\delta(x) \otimes y + \delta(y) \otimes x = y \otimes x \otimes x - x \otimes y \otimes x \\
\sigma(\delta \otimes \mathrm{id})\delta(y) &= x \otimes y \otimes x - x \otimes x \otimes y \\
\sigma^2(\delta \otimes \mathrm{id})\delta(y) &= x \otimes x \otimes y - y \otimes x \otimes x
\end{aligned} \tag{5.29}$$

and see that it is indeed vanishing as well. This example completes our review of the mathematical background and we can now proceed with the deformation of the BMS algebra.

5.3 κ-deformed three-dimensional BMS algebra

In this subsection we present the twist deformation of the three-dimensional BMS algebra $\mathcal{B}^3$ defined in (2.37). It partially reproduces our publications [14, 58]. The physically important example of Hopf algebra that we will use as a starting point here is the κ-Poincaré Hopf algebra [7]-[11], which exists in three different versions characterized by the fixed vector τ. This vector is spacelike, timelike or lightlike and defines the classical r-matrix [59]

$$r_\tau = \tau^\alpha M_{\alpha\mu} \wedge P^\mu \tag{5.30}$$

where $M_{\alpha\mu}, P^\mu$ form a basis of the Poincaré algebra

$$\begin{aligned} \left[M_{\mu v}, M_{\rho\lambda}\right] &= i\left(\eta_{\mu\lambda} M_{v\rho} - \eta_{v\lambda} M_{\mu\rho} + \eta_{v\rho} M_{\mu\lambda} - \eta_{\mu\rho} M_{v\lambda}\right) \\ \left[M_{\mu\nu}, P_\rho\right] &= i\left(\eta_{v\rho} P_\mu - \eta_{\mu\rho} P_v\right) \quad , \quad \left[P_\mu, P_\lambda\right] = 0 \end{aligned} \tag{5.31}$$

and $\eta_{\mu\nu}$ is the Minkowski metric. The Schouten bracket of (5.30) turns out to be

$$[[r_\tau, r_\tau]] = -\eta(\tau,\tau)\Omega, \tag{5.32}$$

where $\Omega = M_{\mu\nu} \wedge P^\mu \wedge P^\nu$ and therefore, out of these three different κ-Poincaré versions, only the lightlike one is coboundary triangular and can thus be constructed by a twist. The deformation of the Poincaré algebra via twist can then be extended to the whole $\mathcal{B}^3$ since from (2.37) we know the commutator of the elements of the Poincaré subalgebra and elements of $\mathcal{B}^3$ and, as will be explained in more detail below, this is sufficient to perform the twist deformation of $\mathcal{B}^3$. Since for the timelike and spacelike cases it is far less obvious how such an extension of the deformation to $\mathcal{B}^3$ could be performed we choose the lightlike case.

The lightlike (also called lightcone) κ-Poincaré deformation has the further advantage that it automatically guarantees that the embeddings of the Poincaré sub-Hopf algebras are consistent in the sense that the new coproduct is a homomorphism for all $X, X' \in U\mathcal{B}^3$ since

$$\Delta(XX') = \mathcal{F}\Delta_0(XX')\mathcal{F}^{-1} = \mathcal{F}\Delta_0(X)\mathcal{F}^{-1}\mathcal{F}\Delta_0(X')\mathcal{F}^{-1} = \Delta(X)\Delta(X'). \tag{5.33}$$

In fact, one can even make a stronger argument that only the coboundary triangular Hopf algebra is possible in $U\mathcal{B}^3$. In the Poincaré algebra there is only one nontrivial candidate for the ad-invariant element in eq.(5.24)

$$\Omega = M_{\mu\nu} \wedge P^\mu \wedge P^\nu \tag{5.34}$$

but this is not ad-invariant in the $\mathcal{B}^3$. Thus the Schouten bracket has to vanish which is the defining relation for a triangular Hopf algebra. Therefore the lightlike deformation is actually the only one that can be constructed for $\mathcal{B}^3$.

To explicitly construct the lightcone κ-deformation one can first canonically assign a Hopf algebra structure to $U\mathcal{B}^3$ by using the undeformed coproduct and antipode defined in eq.(5.7) for every generator X. This trivial coalgebra structure can then be deformed by using a 2-cocycle twist. In lightcone coordinates the Minkowski metric reads $ds^2 = 2dx^+dx^- - dx_1^2$ and we choose a null vector $\tau = (1,0,0)$ such that the r-matrix (5.30) is given by

$$r_\varepsilon = M_{+-} \wedge P^- + \varepsilon M_{+1} \wedge P^1 \tag{5.35}$$

and the twist, which is called extended Jordanian twist, has the following form [59, 60]

$$\mathcal{F}_\varepsilon = \exp\left(-\varepsilon\eta^{11}\frac{i}{\kappa}M_{+1}\otimes P_1\right)\exp\left(-iM_{+-}\otimes\eta^{+-}\log\left(1+\frac{P_+}{\kappa}\right)\right), \tag{5.36}$$

where $\varepsilon = 0, 1$. We introduced the factor ε which can be either 1 corresponding to the full deformation or 0 which leaves only the Jordanian part [60] of the twist.

Since there are in fact infinitely many embeddings of the Poincaré algebra into $\mathcal{B}^3$, see eqs.(2.42)-(2.43), we obtain a whole family of twists parametrized by n. These different embeddings of the Poincaré algebra into $\mathcal{B}^3$ [10] allow for twists that turn out to lead to non-isomorphic Hopf algebras living in the same universal enveloping algebra structure. This means that there is a family of unitary twisting elements $\mathcal{F}_{n,\varepsilon}$ satisfying the two-cocycle and normalization conditions which read

$$\begin{aligned}\mathcal{F}_{n,\varepsilon} &= \exp\left(-\varepsilon\eta^{11}\frac{i}{\kappa}M_{+1}\otimes P_1\right)\exp\left(-iM_{+-}\otimes\eta^{+-}\log\left(1+\frac{P_+}{\kappa}\right)\right)\\ &= \exp\left(\varepsilon\frac{i}{n\kappa\sqrt{2}}l_n\otimes T_0\right)\exp\left(-l_0\otimes 1/n\log\left(1+\frac{i}{\kappa\sqrt{2}}T_n\right)\right).\end{aligned} \tag{5.37}$$

To calculate the deformed coproducts (5.9)

$$\Delta_{n,\varepsilon}(l_m) = \mathcal{F}_{n,\varepsilon}\Delta_0(l_m)\mathcal{F}_{n,\varepsilon}^{-1}, \quad \Delta_{n,\varepsilon}(T_m) = \mathcal{F}_{n,\varepsilon}\Delta_0(T_m)\mathcal{F}_{n,\varepsilon}^{-1} \tag{5.38}$$

we first write $\mathcal{F}_{n,\varepsilon} = e^Ce^D$ and use that $\mathcal{F}_{n,\varepsilon}^{-1} = e^{-D}e^{-C}$. We thus need to evaluate terms of the form

$$\Delta = e^Ce^D\Delta_0e^{-D}e^{-C} \tag{5.39}$$

[10] In four dimensions similar embeddings exist and we start with the three dimensional case mainly for simplicity.

and to this end it is convenient to make use of the the Hadamard formula

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A, ...[A,}_{\text{n times}} B]...].$$

This formula is used twice, first we evaluate $\Delta' = e^D \Delta_0 e^{-D}$ and then in a second step $\Delta = e^C \Delta' e^{-C}$ and in this way obtain the following deformed coproducts

$$\begin{aligned}
\Delta_{n,\varepsilon}(l_m) =& 1 \otimes l_m - i\frac{(m-n)}{n\sqrt{2}\kappa} l_0 \otimes T_{m+n}\Pi_{+n}^{-1} + i\varepsilon \frac{m}{n\sqrt{2}\kappa} l_n \otimes T_m \\
&+ \varepsilon \frac{(m-n)}{n2\kappa^2} l_n \otimes T_0 T_{m+n} \Pi_{+n}^{-1} + l_m \otimes \Pi_{+n}^{\frac{m}{n}} \\
&+ \varepsilon \sum_{k=1}^{\infty} \left(\frac{i}{n\sqrt{2}\kappa}\right)^k \frac{1}{k!} f_{nm}^k l_{m+kn} \otimes T_0^k \Pi_{+n}^{\frac{m}{n}}, \qquad (5.40)\\
\Delta_{n,\varepsilon}(T_m) =& 1 \otimes T_m + T_m \otimes \Pi_{+n}^{\frac{m}{n}} \\
&+ \varepsilon \sum_{k=1}^{\infty} \left(\frac{i}{n\sqrt{2}\kappa}\right)^k \frac{1}{k!} f_{nm}^k T_{m+kn} \otimes T_0^k \Pi_{+n}^{\frac{m}{n}}, \qquad (5.41)
\end{aligned}$$

where

$$f_{nm}^k = \prod_{j=0}^{k-1} (n - (m + jn)), \quad \Pi_{+n} = \left(1 + i\frac{T_n}{\sqrt{2}\kappa}\right). \qquad (5.42)$$

Each coproduct labeled by $n = 1, 2, \ldots$ represents a different Hopf algebra deformation of the enveloping algebra, i.e. a quantum group. Moreover, depending on the value of $\varepsilon = 0$ or $\varepsilon = 1$ one has two quantum group structures for each n: Jordanian or extended Jordanian. Furthermore, each of these quantum groups admits only one Hopf subalgebra, which is for any n spanned by the elements $l_0, l_{\pm n}, T_0, T_{\pm n}$ (to see that they are Hopf subalgebras notice that for $m = n \Rightarrow f_{nm}^k = 0$) and is isomorphic to the (lightlike) κ-Poincaré Hopf algebra, i.e. the case $n = 1$.

The deformed antipode could be obtained from (5.11) but it appears that the resulting expressions can not be written in closed form. We therefore only present the leading order which is most easily obtained from the defining property of the antipode in (5.6) (first equation therein). Considering first T_m and evaluating the right-hand side of (5.6) then yields

$$\begin{aligned}
& m \circ (\mathrm{id} \otimes S_{n,\varepsilon}) \circ \Delta_{n,\varepsilon}(T_m) = 0 \\
\Rightarrow & S_{n,\varepsilon}(T_m) + T_m \left(1 + i\frac{m S_{n,\varepsilon}(T_n)}{n\sqrt{2}\kappa}\right) \\
& \quad + \frac{i\varepsilon}{n\sqrt{2}\kappa}(n-m) T_{m+n} S_{n,\varepsilon}(T_0) + O(\kappa^{-2}) = 0 \\
\Rightarrow & S_{n,\varepsilon}(T_m) = -T_m \left(1 - i\frac{m T_n}{n\sqrt{2}\kappa}\right) + \frac{i\varepsilon}{n\sqrt{2}\kappa}(n-m) T_{m+n} T_0 + O(\kappa^{-2}). \qquad (5.43)
\end{aligned}$$

Similarly we obtain the deformed antipode $S(l_m)$

$$S_{n,\varepsilon}(l_m) = -l_m - \frac{i}{\sqrt{2}n\kappa}\big[(m-n)l_0T_{m+n} \\ - m(l_mT_n + \varepsilon l_nT_m) - \varepsilon(n-m)l_{m+n}T_0\big]. \tag{5.44}$$

Note that if one tried to obtain the deformed antipode in all orders in this way one would in general obtain implicit equations for the antipode.

Each quantum group has as a classical limit the corresponding Lie bialgebra structure. In our case the bialgebra cobrackets are obtained via (5.18) from classical r-matrices of the form

$$r_{n,\varepsilon} = \eta^{-+}M_{+-} \wedge P_+ + \varepsilon\,\eta^{11}M_{+1} \wedge P_1 = \frac{1}{\sqrt{2}n}\left(l_0 \wedge T_n + \varepsilon\, l_n \wedge T_0\right) \tag{5.45}$$

and explicitly read

$$\delta_{n,\varepsilon}(l_m) = \frac{i}{\sqrt{2}n}\big((n-m)l_0 \wedge T_{m+n} - \varepsilon m l_n \wedge T_m \\ - \varepsilon(m-n))l_{m+n} \wedge T_0 + m l_m \wedge T_n\big)\,, \tag{5.46}$$

$$\delta_{n,\varepsilon}(T_m) = \frac{i}{\sqrt{2}n}(\varepsilon(n-m)T_{m+n} \wedge T_0 + mT_m \wedge T_n)\,. \tag{5.47}$$

One can check that this indeed corresponds to the anti-symmetrized $O(\kappa^{-1})$ terms of the coproducts as it should according to (5.15).

5.4 κ-deformed four-dimensional BMS algebra

The deformation of the four-dimensional BMS algebra can be performed completely analogously to the three-dimensional case. The results of this subsection have been published in [14, 58]. Starting point is the Lie algebra $\mathcal{B}^4_{\text{ext}}$, i.e. the algebra in eq. (2.20) with $l, m, n \in \mathbb{Z}$, and for convenience we introduce a new basis via the linear combinations

$$k_m = l_m + \bar{l}_m, \quad \bar{k}_m = -i\left(l_m - \bar{l}_m\right), \\ S_{mp} = \frac{1}{2}\left(T_{mp} + T_{pm}\right), \quad A_{mp} = -\frac{i}{2}\left(T_{mp} - T_{pm}\right) \tag{5.48}$$

in terms of which the algebra $\mathcal{B}^4_{\rm ext}$ takes the form

$$
\begin{aligned}
[k_n, k_m] &= (n-m)k_{n+m}, \quad \left[\bar{k}_n, \bar{k}_m\right] = -(n-m)k_{n+m}, \\
\left[k_n, \bar{k}_m\right] &= (n-m)\bar{k}_{n+m}, \\
\left[k_n, S_{pq}\right] &= \left(\frac{n+1}{2} - p\right) S_{p+n,q} + \left(\frac{n+1}{2} - q\right) S_{q+n,p}, \\
\left[\bar{k}_n, S_{pq}\right] &= \left(\frac{n+1}{2} - p\right) A_{p+n,q} + \left(\frac{n+1}{2} - q\right) A_{q+n,p}, \\
\left[k_n, A_{pq}\right] &= \left(\frac{n+1}{2} - p\right) A_{p+n,q} + \left(\frac{n+1}{2} - q\right) A_{p,q+n}, \\
\left[\bar{k}_n, A_{pq}\right] &= \left(\frac{n+1}{2} - p\right) S_{p+n,q} + \left(\frac{n+1}{2} - q\right) S_{q+n,p}.
\end{aligned}
\tag{5.49}
$$

The elements which span the Poincare subalgebra in lightcone coordinates, with $a, b = 1, 2$

$$
\begin{aligned}
&[M_{+a}, M_{-b}] = -i\left(M_{ab} + \eta_{ab} M_{+-}\right) \quad , \quad [M_{\pm a}, M_{\pm b}] = 0 \\
&[M_{\pm a}, M_{bc}] = i\left(\eta_{ab} M_{\pm c} - \eta_{ac} M_{\pm b}\right) \quad , \quad [M_{+-}, M_{\pm a}] = \pm i M_{\pm a} \\
&[M_{+-}, P_\pm] = \pm i P_\pm \quad , \quad [M_{\pm a}, P_b] = i\eta_{ab} P_\pm \\
&[M_{\pm a}, P_\pm] = [M_{+-}, P_a] = 0 \quad , \quad [M_{\pm a}, P_\mp] = -iP_a
\end{aligned}
\tag{5.50}
$$

are now given by

$$
\begin{aligned}
&k_0 = iM_{+-},\ \bar{k}_0 = -iM_{12},\ k_1 = -i\sqrt{2}M_{+1}, \\
&k_{-1} = i\sqrt{2}M_{-1},\ \bar{k}_1 = i\sqrt{2}M_{+2},\ \bar{k}_{-1} = i\sqrt{2}M_{-2}, \\
&S_{00} = -i\sqrt{2}P_-,\ S_{11} = -i\sqrt{2}P_+,\ S_{01} = iP_1,\ A_{01} = -iP_2.
\end{aligned}
\tag{5.51}
$$

As before the structure we are going to deform is the Hopf algebra obtained by equipping the universal enveloping algebra $U\mathcal{B}^4_{\rm ext}$ with an undeformed coproduct and antipode. Since in the four-dimensional case there are also infinitely many embeddings of the Poincaré algebra into $\mathcal{B}^4_{\rm ext}$, as discussed after eq. (2.26), there exists a family of twisting elements, with $n \in \mathbb{Z}$

$$
\begin{aligned}
\mathcal{F}_{n,\varepsilon} &= \exp\left(-\varepsilon \frac{i}{\kappa} M_{+a} \otimes P_a\right) \exp\left(-iM_{+-} \otimes \log\left(1 + \frac{P_+}{\kappa}\right)\right) \\
&= \exp\left(-\frac{i\varepsilon}{\sqrt{2}\kappa(1-2n)}\left(k_{1-2n} \otimes S_{n,1-n} + \bar{k}_{1-2n} \otimes A_{n,1-n}\right)\right) \\
&\quad \times \exp\left(\frac{k_0}{1-2n} \otimes \log\left(1 + \frac{i}{\sqrt{2}\kappa} S_{1-n,1-n}\right)\right),
\end{aligned}
\tag{5.52}
$$

where the first line is the lightlike κ-Poincaré twisting element given in [59]. The second line has been obtained by first expressing M, P in terms of $k, \bar{k}, S, A$ according to (5.51) and then substituting the embeddings as follows

$$\{l_0, l_{\pm 1}, \bar{l}_0, \bar{l}_{\pm 1} T_{00}, T_{11}, T_{10}, T_{01}\}$$
$$\Rightarrow$$
$$\Big\{l_0, l_{\pm(1-2n)}, \bar{l}_0, \bar{l}_{\pm(1-2n)}, T_{n,n}, T_{1-n,1-n}, T_{1-n,n}, T_{n,1-n}\Big\}\,. \tag{5.53}$$

From the twist (5.52) we obtain the following deformed coproducts

$$\begin{aligned}\Delta_{n,\varepsilon}(k_m) = {} & 1 \otimes k_m + k_m \otimes 1 \\ & - \frac{i}{(1-2n)\sqrt{2}\kappa}\big((m+2n-1)k_0 \otimes S_{1-n+m,1-n} + m k_m \otimes S_{1-n,1-n}\big) \\ & + \frac{i\varepsilon}{2(1-2n)\sqrt{2}\kappa}\Big(k_{1-2n} \otimes [(m-2n+1)S_{n+m,1-n}(m+2n-1)S_{1-n+m,n}] \\ & + \bar{k}_{1-2n} \otimes [(m-2n+1)A_{n+m,1-n} + (m+2n-1)A_{n,1-n+m}] \\ & - 2(1-2n-m)[k_{1-2n+m} \otimes S_{n,1-n} + \bar{k}_{1-2n+m} \otimes A_{n,1-n}]\Big) + O(\kappa^{-2})\end{aligned} \tag{5.54}$$

$$\begin{aligned}\Delta_{n,\varepsilon}(\bar{k}_m) = {} & 1 \otimes \bar{k}_m + \bar{k}_m \otimes 1 \\ & - \frac{i}{(1-2n)\sqrt{2}\kappa}\big((m+2n-1)k_0 \otimes A_{1-n+m,1-n} + m k_m \otimes A_{1-n,1-n}\big) \\ & + \frac{i\varepsilon}{2(1-2n)\sqrt{2}\kappa}\Big(k_{1-2n} \otimes [(m-2n+1)A_{n+m,1-n} \\ & + (m+2n-1)A_{1-n+m,n}] \\ & + \bar{k}_{1-2n} \otimes [(m-2n+1)S_{n+m,1-n} + (m+2n-1)S_{n,1-n+m}] \\ & - 2(1-2n-m)[\bar{k}_{1-2n+m} \otimes S_{n,1-n} - k_{1-2n+m} \otimes A_{n,1-n}]\Big) + O(\kappa^{-2})\end{aligned} \tag{5.55}$$

$$\begin{aligned}\Delta_{n,\varepsilon}(S_{pq}) = {} & 1 \otimes S_{pq} + S_{pq} \otimes 1 \\ & - \frac{i}{(1-2n)\sqrt{2}\kappa}(p+q)S_{pq} \otimes S_{1-n,1-n} \\ & + \frac{i\varepsilon}{(1-2n)\sqrt{2}\kappa}\Big([(1-n-p)S_{p+1-2n,q} + (1-n-q)S_{q+1-2n,p}] \otimes S_{n,1-n} \\ & + [(1-n-p)A_{p+1-2n,q} \\ & \quad + (1-n-q)A_{q+1-2n,p}] \otimes A_{n,1-n}\Big) + O(\kappa^{-2})\end{aligned} \tag{5.56}$$

$$
\begin{aligned}
\Delta_{n,\varepsilon}(A_{pq}) = 1 \otimes A_{pq} + A_{pq} \otimes 1 \\
&- \frac{i}{(1-2n)\sqrt{2}\kappa}(p+q)A_{pq} \otimes S_{1-n,1-n} \\
&+ \frac{i\varepsilon}{(1-2n)\sqrt{2}\kappa}\Big([(1-n-p)A_{p+1-2n,q} + (1-n-q)A_{p,q+1-2n}] \otimes S_{n,1-n} \\
&+ [(1-n-p)S_{p+1-2n,q} \\
&+ (1-n-q)S_{q+1-2n,p}] \otimes A_{n,1-n}\Big) + O(\kappa^{-2}).
\end{aligned} \tag{5.57}
$$

For each n there exists only a single Hopf subalgebra spanned by the set of ten generators $k_0, k_{1-2n}, \bar{k}_0, \bar{k}_{1-2n}, S_{n,n}, S_{1-n,1-n}, S_{n,1-n}, S_{1-n,n}, A_{n,1-n}, A_{1-n,n}$. These subalgebras are isomorphic to the Poincaré subalgebra, i.e. the case $n = 0$. To see that these generators indeed form subalgebras consider for instance $\Delta(k_m)$ and for $m = 0$ it is directly clear that it only contains elements from the aforementioned set of ten generators. For $m = 1-2n$ the only remaining contribution comes from terms proportional to $S_{n+m,1-n} = S_{1-n,1-n}$ and is thus also in the set and similarly one can check the remaining coproducts.

The Lie bialgebra cobracket can be obtained simply by anti-symmetrizing the leading order expressions for the coproducts (5.54)-(5.57) and we therefore do not write them here explicitly. The expressions for the deformed antipode can be found in the Appendix (8.1).

5.5 Physical interpretation of coproduct and antipode

So far our treatment of the deformation of the BMS algebra in three and four dimensions has been mainly mathematical and we can now turn to explaining how the coproduct and antipode can be interpreted physically.

The role of the coproduct is to define how an operator, e.g. the momentum operator, acts on a two-particle state to obtain the total momentum, see for example [18]. In the undeformed case this just yields the familiar composition rule of momenta

$$
\begin{aligned}
P_{\text{tot}} |P^1\rangle \otimes |P^2\rangle &= \Delta(P) |P^1\rangle \otimes |P^2\rangle \\
&= (1 \otimes P + P \otimes 1) |P^1\rangle \otimes |P^2\rangle \\
&= (P^1 + P^2) |P^1\rangle \otimes |P^2\rangle
\end{aligned} \tag{5.58}
$$

but if we, for example, consider the deformed coproduct (5.41) this composition rule is modified

$$
\Delta_{n,0}(T_m) |T^1\rangle \otimes |T^2\rangle = (T_m^1 + T_m^2 + \frac{im}{n\sqrt{2}\kappa} T_m^1 T_n^2) |T^1\rangle \otimes |T^2\rangle \tag{5.59}
$$

and it is common to use the notation

$$
T_m^1 \oplus T_m^2 = T_m^1 + T_m^2 + \frac{im}{n\sqrt{2}\kappa} T_m^1 T_n^2. \tag{5.60}
$$

We can therefore understand the coproduct as the structure that defines a deformed summation rule and the antipode can be understood as the compatible deformed subtraction and will be denoted as $\ominus$. Using the expression for the antipode (5.43), with $\varepsilon = 0$, we can write in leading order

$$T_m \oplus (\ominus T_m) = T_m \oplus (-T_m + \frac{im}{n\sqrt{2}\kappa} T_m T_n) = 0, \tag{5.61}$$

which demonstrates aforementioned compatibility.

An interesting feature of the deformed coproducts in four dimensions (5.54)-(5.57) is that it leads to a modified composition rule such that the addition of two supermomenta, i.e. momenta outside of the Poincaré sector, in general will contain Poincaré momenta.[11] The deformed coproduct $\Delta_{1,0}(A_{pq})$ in eq. (5.57), for instance, contains the Poincaré momenta $S_{0,0}, S_{0,1}$ and $A_{0,1}$. The same is true for the addition of two superrotations which will generically contain an element k_0 or $\bar{k}_0$ from the Poincaré sector. This fact is going to play an important role in section 6.3, where we explain how it could enter an ongoing discussion concerning the black hole information paradox [22].

[11]This is also true for the three-dimensional case, but we are interested here in the physical, four-dimensional case.

6 Black hole entropy and information loss paradox

6.1 Introduction

The discovery that black holes carry an entropy proportional to their horizon area A according to the celebrated Bekenstein-Hawking formula

$$S = \frac{A}{4\hbar G}\,, \tag{6.1}$$

is now more than forty years old [21, 61]. The physical picture is that due to quantum effects a black hole of mass M evaporates [21] and it does so in such a way that the radiation it emits has an exact thermal spectrum at the Hawking temperature

$$T_H = \frac{\hbar}{8\pi GM}. \tag{6.2}$$

Despite numerous derivations of the entropy-area relation (6.1) existing in a variety of approaches to quantum gravity (see [62] for a comprehensive listing), the fundamental question concerning the nature of the degrees of freedom responsible for such entropy has not yet found a conclusive answer.

In subsection 6.2 we revisit [65] one of the earliest attempts at addressing this question by 't Hooft [12], in which quanta of a field in thermal equilibrium at the Hawking temperature near the horizon are considered as possible candidates for the origin of the Bekenstein-Hawking entropy. The main novelty of our treatment is the inclusion of back-reaction effects on the spacetime metric due to the evaporation of the black hole which allows us to give a natural explanation for a certain regulator, the so-called brick wall, 't Hooft had to introduce ad hoc. The brick wall is situated just above the event horizon and it is assumed that all fields below it are vanishing in order to cure divergences that would appear otherwise.

After completing our discussion of black hole entropy we turn in subsection 6.3 to another far reaching consequence of the existence of black hole evaporation, namely the information loss paradox [22]. We will briefly introduce what the paradox is about and explain a possible loophole in the derivation of [22] which has recently been proposed by Hawking, Perry and Strominger [13]. It is intimately connected with the BMS symmetry and is of semi-classical nature, i.e. spacetime is considered to be classical and matter fields quantum. Not long after, Bousso, Mirbabayi and Porrati [63, 64] have put forward a counter argument which they use to claim that the BMS symmetry has in fact no bearing on the information paradox whatsoever. In yet another twist to that story we argue that the κ-deformation of the BMS symmetry potentially invalidates one of the core assumptions of [64], so that their argument would no longer be an obstruction to the arguments of [13]. This line of argument would suggest that a semi-classical approach is not sufficient to resolve the information loss paradox but quantum gravity effects are necessary.

6.2 Brick wall entropy

Since due to quantum effects black holes radiate thermally [21], one of the earliest attempts, by 't Hooft [12], at explaining the microscopical degrees of freedom underlying black hole entropy focused on the quanta of a field in thermal equilibrium at the Hawking temperature near the horizon. As it turns out the counting of modes needed for deriving the thermodynamic partition function of the field yields a divergent result due to an infinite contribution coming from the black hole horizon. 't Hooft noticed that introducing a crude regulator by requiring the vanishing of the field at a small radial distance from the horizon allows to obtain a finite horizon contribution to the entropy proportional to the area. Appropriately tuning the distance of such a "brick wall" from the horizon one can exactly reproduce the Bekenstein-Hawking formula (6.1). This result, albeit suggestive, replaces the question about the origin of Bekenstein-Hawking entropy with a question about the nature of the brick wall boundary.

In this section, which partially reproduces our publication [65], we study the effect of backreaction on the field propagating in the vicinity of the black hole horizon. We do this by replacing the usual Schwarzschild metric by a dynamic, "evaporating" metric first proposed in [66], in which the effects of backreaction are parametrized by the luminosity of the radiating black hole. After solving the field equations in such a metric we proceed to the usual mode counting for the field. The key feature of our model is that the small luminosity creates a "quantum ergosphere", a region between the apparent horizon and the event horizon which effectively acts as a brick wall providing a finite horizon contribution to the entropy. As we show below, within the small luminosity and quasi-static approximations we use we are able to reproduce the Bekenstein-Hawking result within very good accuracy.

Our starting point is the result by Bardeen [66] (see also [67] and [68]) that the metric of a spherically symmetric black hole slowly emitting Hawking radiation has the following form

$$ds^2 = -e^{2\psi}\left(1 - \frac{2m}{r}\right)dv^2 + 2e^{\psi}dvdr + r^2 d\Omega, \tag{6.3}$$

where ψ and m are functions of the advanced time v and the radial coordinate r. Einstein field equations relate ψ and m to the energy-momentum tensor $T_{\mu\nu}$ by

$$\frac{\partial m}{\partial r} = 4\pi r^2 T_v^v \tag{6.4}$$

$$\frac{\partial m}{\partial v} = 4\pi r^2 T_v^r \tag{6.5}$$

$$\frac{\partial \psi}{\partial r} = 4\pi r e^{\psi} T_r^v \tag{6.6}$$

and as is explained in [67] we also assume that near the horizon the components of T_{vv}, T_{rv} are of order L/r^2. For m constant and $\psi = 0$ this metric reduces to the Schwarzschild one, while for $\psi = 0$ and $m = m(v)$ it becomes the Vaidya metric. Following [67] we define the mass of the black hole at a given time to be $M(v) = m(v, r = 2m)$ and its luminosity to be $L = -\frac{dM}{dv}$.

We work in the regime of small luminosity[12] $L \ll 1$ and in what follows we will focus on the near-horizon features of the metric (6.3). To this end we introduce a new "co-moving" radial coordinate $\rho = r - 2M = r - 2M_0 + 2Lv$ and assume that ρ is small, of the same order as Lv, so that in our computations we will only keep terms which are at most linear in ρ and L. Further, the Einstein equations allow us to set, for convenience, $\psi(r = 2M) = 0$, which in our approximation makes the function ψ disappear from all the linearized expressions. Indeed

$$\psi(r) \simeq \psi(r = 2M) + \left.\frac{\partial\psi}{\partial r}\right|_{r=2M} \rho \tag{6.7}$$

and it follows from Einstein equations that $\partial\psi/\partial r \sim L/r$ at $r = 2M$, so that the first term in (6.7) vanishes, while the second is of higher order and can be neglected. A similar argument can be applied to m

$$m(v, r) \simeq m(v, r = 2M) + \left.\frac{\partial m}{\partial r}\right|_{r=2M} \rho \tag{6.8}$$

and since $\partial m/\partial r \sim L$ we can write to first order in perturbation theory

$$m(v, r) \simeq M(v) \simeq M_0 - Lv\,. \tag{6.9}$$

In terms of the co-moving radial coordinate the metric near $r = 2M$ takes the form

$$ds^2 = -\left(\frac{\rho}{\rho + 2M} + 4L\right) dv^2 + 2dvd\rho + (\rho + 2M)^2 d\Omega. \tag{6.10}$$

The metric (6.3) has several horizon-like structures. We first consider the apparent horizon (AH), defined as the outermost trapped surface, i.e. the surface from which no light ray can move outwards. One characterizes this feature with the help of the expansion Θ of a congruence of null geodesics, which describes the fractional change of the congruence's area. The apparent horizon is defined as a surface for which $\Theta = 0$.

The expansion Θ of a congruence of null geodesics is given by [67]

$$\Theta = l^{\mu}{}_{;\mu} - \kappa\,, \tag{6.11}$$

where l_μ is the tangent vector field to the congruence, $\kappa = -\beta^\mu l^\nu l_{\mu;\nu}$ and β_μ is an auxiliary null vector which fulfills $l_\mu\beta^\mu = -1$. This auxiliary vector appears since in the case of a null curve it is at first not clear how to define a subspace normal to the tangent vector of this curve, because a null vector is normal to itself. Such a subspace is crucial for the description of geodesic congruences, however, and for null geodesics it is defined to consist

[12] Since $L \propto 1/M^2$, for a black hole with solar mass the value would be $L \sim 10^{-76}$ in Planck units.

of all those vectors which are orthogonal to both l^μ and β^μ. Choosing the null vectors

$$l^\mu = (l^v, l^r) = \left(1, \frac{1}{2}\left(1 - \frac{2M}{r}\right)\right), \; \beta^\mu = (0, -1), \tag{6.12}$$

we obtain the following expressions for the expansion in Bardeen coordinates (r, v) and co-moving coordinates (ρ, v)

$$\Theta(r) = \frac{M}{r^2} - \frac{1}{4M}, \quad \Theta(\rho) = \frac{M}{(\rho + 2M)^2} - \frac{1}{4M}, \tag{6.13}$$

which shows that the apparent horizon is located in the two coordinate systems at $r_{AH} = 2M$ and at $\rho_{AH} = 0$.

In order to capture another horizon-like structure present in the problem, York [67] gives a working definition of what we will call York event horizon (YEH), which lacks the teleological property of the event horizon and is instead based on the local condition

$$\frac{d^2 r}{dv^2} = 0, \tag{6.14}$$

i.e. it characterizes the YEH as the surface at which photons are "stuck". According to this definition the YEH in the Bardeen and co-moving coordinates lies at $r_{YEH} = 2M - 8ML$ and $\rho_{YEH} = -8ML$.

The region between York event horizon and apparent horizon was dubbed by York *quantum ergosphere* [67], and he argues that its presence is an irreducible property of an evaporating black hole.

We will use this observation to shed new light on the brick wall calculation of 't Hooft [12] by including the contributions due to the backreaction, here modeled by a small luminosity L.

The original result of 't Hooft is that the free energy of a scalar field living outside the Schwarzschild black hole has a horizon contribution given by

$$F = -\frac{2\pi^3}{45h}\left(\frac{2M}{\beta}\right)^4 + \dots \tag{6.15}$$

where β is the inverse Bekenstein-Hawking temperature and h is a small cut-off parameter with dimensions of length. From (6.15), using standard manipulations, one can calculate the thermodynamic entropy associated to the field and the resulting contribution from the horizon term above is proportional to the area of the black hole thus qualitatively reproducing the Bekenstein-Hawking entropy-area relation. As we will see, it is a consequence of finite luminosity that the brick-wall thickness h, which is arbitrary in the original 't Hooft calculation, can be now naturally identified with the distance between the event and apparent horizon. Therefore the *quantum ergosphere* [67], the region between the AH and the YEH, plays the role of a physically motivated brick wall.

In order to proceed with the counting of modes of the field we start by solving the equations of motion describing a scalar field in the vicinity of the black hole horizon in the co-moving coordinates introduced earlier. A massless scalar field ϕ in this geometry with metric (6.10) obeys the following equation of motion

$$\left(4L + \frac{\rho}{\rho + 2M}\right) \partial_\rho^2 \phi + 2\frac{\partial_\rho \phi}{\rho + 2M}\left(1 - \frac{M}{\rho + 2M} + 2L\right) + 2\partial_\rho \partial_v \phi + \frac{2}{\rho + 2M}\partial_v \phi - \frac{l(l+1)}{(\rho + 2M)^2}\phi = 0. \tag{6.16}$$

Since we are interested only in the contribution coming from the vicinity of the horizon in the case of small luminosity assuming $\rho/M_0 \ll 1$, $L \ll 1$ and using the quasi-static approximation $\frac{Lv}{2M_0} \ll 1$[13], we can write

$$\frac{2}{\rho + 2M} \approx \frac{1}{M_0}\left(1 - \frac{\rho}{2M_0} + \frac{Lv}{M_0}\right) \tag{6.17}$$

$$\frac{2}{\rho + 2M}\left(1 - \frac{M}{\rho + 2M} + 2L\right) \approx \frac{1}{2M_0}\left(1 + 4L + \frac{Lv}{M_0}\right) \tag{6.18}$$

and equation (6.16) becomes

$$\left(4L + \frac{\rho}{2M_0}\right) \partial_\rho^2 \phi + \frac{1}{2M_0}\left(1 + 4L + \frac{Lv}{M_0}\right) \partial_\rho \phi + 2\partial_\rho \partial_v \phi + \frac{1}{M_0}\left(1 - \frac{\rho}{2M_0} + \frac{Lv}{M_0}\right) \partial_v \phi - \frac{l(l+1)}{(2M_0)^2}\phi = 0. \tag{6.19}$$

We now make use of the standard WKB ansatz

$$\phi(\rho, v) = U(\rho) e^{-i\omega v} e^{i\int^\rho k(\rho')d\rho'}. \tag{6.20}$$

and find that the real part of (6.19) takes the form

$$\left(4L + \frac{\rho}{2M_0}\right)(U'' - k^2 U) + \frac{U'}{2M_0} + 2\omega k U - \frac{l(l+1)}{(2M_0)^2}U = 0, \tag{6.21}$$

which, as a consequence of our approximation scheme, is v-independent. This part is sufficient for obtaining the wavenumber k. The imaginary part could be used to compute the amplitude but since we are only interested in counting the modes of the field with the help of the wavenumber we can ignore it. Moreover the v-independence of (6.21) shows that for the purpose of the entropy computation, to be presented below, the geometry is static. In the WKB approximation one assumes that the amplitude $U(\rho)$ varies slowly

[13]This approximation means that we only consider v which are small compared to the expected lifetime of the black hole $M_0 L^{-1}$.

compared to the wave number

$$\frac{U'}{U} \ll k, \quad \frac{U''}{U} \ll k^2, \tag{6.22}$$

and therefore (6.21) becomes

$$-\left(4L + \frac{\rho}{2M_0}\right) k^2 + 2\omega k - \frac{l(l+1)}{(2M_0)^2} = 0, \tag{6.23}$$

which can be solved for k giving

$$k^{\pm} \approx \frac{\omega \pm \sqrt{\omega^2 - \left(4L + \frac{\rho}{2M_0}\right) \frac{l(l+1)}{(2M_0)^2}}}{4L + \frac{\rho}{2M_0}}, \tag{6.24}$$

where again we neglected the terms which are of higher order in our approximation scheme. These two solutions correspond to incoming and outgoing modes, respectively and can be used to calculate the thermodynamic entropy associated to the field via a count of its number of modes and the derivation of the statistical partition function.

By approximating the sum over l with an integral the number of modes with frequency up to ω is given by, see also [69]

$$g(\omega) = \int_0^{l_{max}} \nu(l, \omega)(2l + 1)dl, \tag{6.25}$$

where $\nu(l, \omega)$ is the number of nodes in the mode with (l, ω) and l_{max} is the maximal value for l such that the square root in (6.24) is real. The quantity ν can be explicitly calculated by considering the modes (6.24) in the box of the radial length Λ, which acts as an infrared regulator

$$\Lambda = \nu \frac{\lambda}{2} = \nu \frac{\pi}{k} \to \pi\nu = \Lambda k, \quad k = \frac{2\pi}{\lambda}, \tag{6.26}$$

where λ is the wavelength of the mode.

In the original brick wall calculation [12] it is assumed that the scalar field, whose entropy we are going to compute, vanishes beyond the brick wall, situated at a small distance h from the Schwarzschild black hole horizon at r_{Sch}, so that all the relevant integrals have the lower limit at $r_{\text{Sch}} + h$. In the case of the Schwarzschild black hole considered in [12] the apparent and event horizon coincide, $r_{\text{Sch}} = r_{EH} = r_{AH}$, however in our case they are different and we must decide at which of the two we impose the scalar field boundary conditions. Our argument relies on the observation that in the brick wall picture the scalar field is to be in thermal equilibrium at temperature T which is identified with the temperature of Hawking radiation. However, by invoking the so-called tunneling picture, it can be argued that the Hawking radiation originates at the vicinity of the apparent, not the event horizon (see [70] and references therein). Thus, remembering that the apparent horizon corresponds to $\rho = 0$, we choose the integration range in the formula above to go from 0 to Λ, where Λ is the infra-red cutoff introduced before, whose explicit value will not interest us here, since the expression for the area contribution to the entropy does not depend on it. The number of nodes is thus given by the integral

$$2\pi\nu(l,\omega) = \int_0^\Lambda k^+ d\rho = \int_0^\Lambda \frac{\omega + \sqrt{\omega^2 - \left(4L + \frac{\rho}{2M_0}\right)\frac{l(l+1)}{(2M_0)^2}}}{4L + \frac{\rho}{2M_0}} d\rho, \tag{6.27}$$

where we used eq. (6.24) and only considered the contribution from the outgoing modes. We notice that the ingoing solutions close to the apparent horizon are moving towards the singularity and one can argue that they can not contribute to the entropy since they do not appear outside of the apparent horizon where we are counting the modes. As we will find below, this choice is also justified a posteriori, by the remarkable agreement of our final result with the Bekenstein-Hawking entropy relation.

Let us notice that the equation for the number of nodes above differs from the one obtained previously in the literature in [69] in two aspects. First, due to the approximations we made there is no dependence on the advanced time v and second, we do not have to introduce a cut-off close to the horizon, since the finite luminosity prevents the integrand from diverging at $\rho = 0$. The integration with respect to l in eq. (6.25) is taken over those values for which the square root is real and yields

$$g(\omega) = \int_0^\Lambda \frac{5(2GM_0 + \rho)^4\omega^3}{6\pi(8M_0L + \rho(1+4L))^2} d\rho. \tag{6.28}$$

The leading contributions in the integral in (6.28) are thus given by

$$g(\omega) = \frac{5\omega^3 M_0^3}{3\pi L} + \frac{5\omega^3\Lambda^3}{18\pi(1+8L)}\,, \tag{6.29}$$

where the second term is the usual volume contribution and has no relevance for our discussion. The thermodynamic partition function of the field is given by

$$Z = e^{-\beta F}\,, \tag{6.30}$$

where F is the free energy

$$\pi\beta F = \int dg(\omega)\ln\left(1 - e^{-\beta\omega}\right). \tag{6.31}$$

Using (6.29) and neglecting the volume contribution to $g(\omega)$ we have

$$F = \frac{1}{\beta}\int_0^\infty \ln(1 - e^{-\beta\omega})\frac{dg(\omega)}{d\omega} d\omega = -\frac{M_0^3\pi^3}{9L\beta^4}\,, \tag{6.32}$$

from which we can calculate the entropy of the field associated to the horizon boundary

$$S = \beta^2 \frac{\partial}{\partial \beta} F = \frac{4M_0^3\pi^3}{9L\beta^3}. \tag{6.33}$$

Comparing our result for the free energy (6.32) with the standard result obtained from the brick wall calculation (6.15) we see that the brick wall width parameter h introduced by 't Hooft can be expressed in terms of the luminosity of the black hole as

$$h = \frac{32}{5} L M_0 \,, \tag{6.34}$$

and thus the backreaction of the quantum radiance on the horizon structures of the black hole naturally provides the regulator needed for a finite horizon contribution to the field entropy.

In order to obtain an expression for the entropy (6.33) to be compared to the Bekenstein-Hawking relation (6.1) we now have to spell out the explicit form of the luminosity L in terms of the black hole mass M_0. In the first order approximation used in our calculation the luminosity L is a small quantity so that we can identify it with the luminosity L_S of Hawking radiation in the case of a Schwarzschild black hole. Any correction to L_S coming from backreaction effects would be subleading and can therefore be safely neglected. To find L_S, one considers [71] a flux X of radiation with energy ω_k

$$X(\omega_k) = \frac{\Gamma(\omega_k)}{2\pi(e^{8\pi M_0 \omega_k} - 1)} \,, \tag{6.35}$$

where the factor Γ models the backscattering. Integrating the flux times the energy ω we find the luminosity that escapes to infinity

$$L_S = \int_0^\infty d\omega \, \omega X(\omega). \tag{6.36}$$

The factor Γ can be approximated by [72]

$$\Gamma \approx 27\pi M_0^2 \omega^2 \tag{6.37}$$

and integration over ω yields

$$L_S \approx \frac{1.69}{7680\pi M_0^2} \,, \tag{6.38}$$

Plugging the expression (6.38) in (6.33) we finally obtain

$$S = 0.987 \cdot 4\pi M_0^2 = 0.987 S_{BH} \tag{6.39}$$

where S_{BH} is the Bekenstein-Hawking entropy. We thus see that our model reproduces the exact result of Bekenstein-Hawking with an accuracy close to 99%, which is a remarkable result given the rather crude approximations that we used.

There is an important comment to be made at this point. In our calculation of the entropy we assumed that the luminosity of the black hole results from a single massless scalar mode, the same that we used to compute the entropy. Since in nature we do not know any massless scalar field to exist it might be argued that one should use instead in our computations the massless fields that we know about, namely photons and gravitons, i.e., four massless degrees of freedom. Each degree of freedom will contribute the amount (6.33) to the entropy. As for the luminosity one can use the numerical results of Page [73], to see that the contribution to luminosity of photons and gravitons is of order $3\times10^{-5}\,1/M_0^2$ as compared to $7\times10^{-5}\,1/M_0^2$ given by (6.38). This means that the final entropy will be by a factor 8 larger in the case of photons and gravitons than it is in the case of a single scalar. On the other hand it is believed that the Bekenstein–Hawking entropy is fundamental, capturing some essential features of space-time and from that perspective it is hard to imagine that it could depend on the number of massless degrees of freedom in nature, which seems to be rather contingent. The fact that employing a single massless degree of freedom reproduces the correct value, with a small error, indicates that there might be something special about the single massless scalar field model.

To summarize, in this section we showed how small backreaction effects can be introduced in the derivation of the thermodynamic entropy of a field in thermal equilibrium in the proximity of a black hole horizon. The resulting changes due to a small but non-vanishing luminosity on the horizon structure of the black hole provide a natural brick wall regulator for the near-horizon modes of the field. Using the small luminosity and quasi-static approximations we were able to solve the equations of motion for a scalar field in the evaporating metric to find an explicit expression for the field modes, the degrees of freedom contributing to the thermodynamic partition function of the field. We showed that once the width of the quantum ergosphere is set by the Hawking luminosity the horizon contribution to the entropy of the field is in very good agreement with the Bekenstein-Hawking relation for the black hole entropy. In the original brick wall calculation the width of the brick wall had to be adjusted by hand in order to have the correct proportionality factor between entropy and the black hole area. From this point of view we find our result particularly suggestive: the non-trivial horizon geometry determined by the backreaction of the Hawking flux leaves no arbitrary parameter to be tuned to obtain the desired result.

6.3 Information loss paradox and BMS

At the end of Section 5.5 we have mentioned that the κ-deformation of BMS can be argued to affect a discussion concerning the information loss paradox. To properly explain our argument we first take a moment to review the main points of this paradox, a loophole which potentially leads to a way out of the paradox [13] involving the BMS symmetry and a counter argument aiming to invalidate this loophole [63, 64].

By means of a semi-classical derivation Hawking [21] demonstrated that black holes radiate particles and found that the emission is described by an exact blackbody spectrum. It has furthermore been argued by Hawking [22] that this radiation does not carry any information about the matter fields that have formed the black hole. The microscopic origin of the radiation are particle-antiparticle pairs which form in the vicinity of the black hole horizon with particles of negative energy falling into the black hole and particles with positive energy escaping to infinity. The important point is that this process is completely independent of the interior of the black hole and the radiation therefore carries no information about the infalling matter whatsoever. This point is particularly stressed in [74], where spacelike slices of the black hole spacetime are considered and it is shown that on such a slice any infalling matter is extremely far away from the place where the particle pairs are created. Due to the no-hair theorem [75] it is furthermore usually assumed that the exterior of the black hole carries very little information, namely only the total mass, angular momentum and electric charge. One can therefore argue that the Hawking radiation can at most carry information about these three quantities.

Eventually the black hole will decay to the vacuum,[14] which is usually considered to be unique. It would therefore appear that all information falling into the black hole, for instance arbitrary multipole momenta the infalling matter carries, is lost because it can neither be stored in the geometry of spacetime nor in the Hawking radiation. On the other hand, quantum mechanics, which has been used to derive this result, demands that the evolution of all matter has to be unitary, i.e. information preserving. This contradiction is, in essence, what is referred to as the information loss paradox.

In [13] Hawking, Perry and Strominger argue that at least one of the assumptions underlying the argument of [22] is in fact flawed as a consequence of the presence of the BMS symmetry. This assumption is that black holes have "no hair" in the sense that externally they can be described using just three numbers, namely the total mass, angular momentum and electric charge. It is pointed out that this statement is true only up to diffeomorphisms, for instance a boosted black hole will additionally carry momentum charge. Since the boosted black hole has a different charge it should be considered as a physically inequivalent state. Previously it has been assumed that the maximum number of such "diffeomorphism hair" is eleven, ten Poincaré charges and electric charge, and that this number is far too low to carry the lost information and can have no bearing on the paradox. When a black hole is supertranslated, however, it will potentially carry an infinite number of superrotation charges [77] which are referred to as "soft hair"[15]. Just as boosting a black hole creates a distinguishable solution so does supertranslating one. This can be seen from the action of the Lie derivative $\delta_f g = \mathcal{L}_f g$ on N_A (as defined in

[14] Occasionally the formation of a remnant is discussed in the literature, see [76] for an overview.

[15] This name is derived from a close relation between the BMS symmetry and soft photon theorems [78]. Soft photons are photons with zero energy and hard ones have finite energy.

(2.9) and with $\xi_f = \xi(f, 0)$ from (2.12))

$$\delta_f N_A = -3 m_B \partial_A f \tag{6.40}$$

and from (2.29) it follows that the supertranslated black hole indeed carries non-vanishing superrotation charge. Since $f(x^A)$ is a general function on the sphere there can be an infinite amount of such charges. Notice that the supertranslated black hole does not carry any supertranslation charge, i.e. "supermomentum", which is analogous to the fact that translating a black hole does not create any momentum and follows from

$$\delta_f \, m_B = 0. \tag{6.41}$$

Now, if the black hole evaporates superrotation charge will be carried away from it by the quanta composing the Hawking radiation but the total amount of charge has to be conserved. This charge conservation enforces an exact correlation between the Hawking quanta of early and late radiation. Consider, for instance[16], an initially stationary black hole and early quanta that carry away total momentum $\mathbf{P}$. Momentum conservation dictates that the remaining black hole has momentum $-\mathbf{P}$ and late quanta will carry the same total momentum. The same argument can be made for the conservation of soft hair. This introduces correlations between early and late quanta which would lead to a deviation from the thermal spectrum and potentially renders the evaporation process unitary.

This argument was refuted by Bousso, Mirbabayi and Porrati [63, 64] who argue that soft hair could not resolve the black hole unitarity problem, because their conservation laws are automatically satisfied. This is shown using a canonical transformation which makes it apparent that the soft modes carrying the superrotation charges decouple from the hard ones, which carry the Poincaré charges, and evolve trivially. As a result only the Poincaré charges evolve non-trivially and each soft charge stays the same all the way through from $\mathcal{I}^-$ to $\mathcal{I}^+$, such that only the conservation of Poincaré charges introduces correlation between early and late quanta in the Hawking radiation. Staying within the metaphor one can say that although a black hole has infinitely many soft hair, they are perfectly combed, so that they do not tangle with the hard ones.

The presence of deformation changes this qualitative picture considerably, as we have suggested in [58], the reason being the co-product structure and the associated modification of the composition laws. Indeed it follows from the discussion in section 5.5 that the addition of two superrotation charges generically depends on the Poincaré charges. Thus, in the above discussed conservation laws the total superrotation charge of the soft modes will generically depend on the Poincaré charge carried by the hard modes. This makes the clear separation of hard and soft part impossible, which is crucial for the argument put forward by [63, 64]. Notice that this effect is negligible at the LHC energy scale, since it is suppressed by $1/\kappa$, which is expected to correspond to the Planck scale.

[16]This example is taken from [63]

We therefore find that the fundamental property of Hopf algebras, the presence of a non-trivial coproduct structure, opens the possibility that early and late Hawking radiation quanta are correlated after all. In respect to the information loss paradox this is only a tentative argument at this state, since it is not clear whether the magnitude of this effect is sufficient to introduce correlations which render the evaporation process unitary.

7 Conclusions

In this thesis we have investigated the consequences of the generalization of the Poincaré to the BMS symmetry in a number of different contexts. We first explained how this generalization arises from the analysis of asymptotic symmetries of spacetime. The key point was that demanding only the asymptotic form of spacetime to be invariant under symmetry transformations, instead of the spacetime as a whole, leads to a much larger, in fact infinite-dimensional, symmetry group. We have also shown that the corresponding symmetry algebra contains infinitely many Poincaré subalgebras, in three and in four dimensions.

In section 3 we have analyzed the asymptotic symmetries of asymptotically flat spacetimes in the Hamiltonian formulation of GR. In contrast to previous treatments we have expressed the asymptotic expansion of the spatial metric and conjugate momenta in terms of a Bondi-type spacetime metric using a 3+1 decomposition. An important insight of this procedure was that the falloff conditions on the momenta translate to the requirement that only such spacetimes are admissible that radiate a finite amount of energy. We then found that for the class of spacetimes we consider the falloff conditions are sufficient to remove all divergences in the theory and there is no need to introduce parity conditions.

The charges which are generating supertranslations are parametrized by two arbitrary functions on the sphere and are finite for every mode of the two functions. The supertranslation sector is therefore larger than the one of the BMS algebra, which is parametrized by a single arbitrary function on the sphere.

A result that remains to be understood better is that spatial translations do not preserve the Bondi determinant condition in our treatment. Why is it that at null infinity this condition is fulfilled automatically whereas at spatial infinity it turns out to be too rigid? Another intriguing question in this context is whether the supertranslation sector at null infinity can be enlarged by dropping the determinant condition. Possibly this enlarged algebra at null infinity is isomorphic to the one we found at spatial infinity? We hope to address these questions in future work.

The topic of section 4 was the construction of a gauge theory in three dimensions with BMS instead of Poincaré as gauge group. In a strict sense this can not be done because the BMS group does not contain a central element, which is needed for the construction of a gauge invariant action. Instead we have used the central element of the Poincaré group, which was motivated by the observation that the resulting action generically contains gauge fields from outside the Poincaré sector. This could potentially result in an interesting extension of the field equations. But it turned out that in order for this action to be gauge invariant these extra gauge fields have to be assumed to vanish, so that one is left with just the standard Poincaré action. Furthermore, we found that this action is invariant under supertranslations, but this invariance is trivial in the sense that the gauge fields themselves are invariant, with the exception of two modes of supertranslations.

We have also considered the case of a finite cosmological constant, so that the gauge group describes the asymptotic symmetry of spacetimes which asymptotically approach Anti-de Sitter spacetime. The main difference to the asymptotically flat case is that supertranslations do not commute anymore. Again, the considered algebra has no central element and we construct the action using the one from the isometry group of Anti-de Sitter spacetime. To be gauge invariant the resulting action can only contain gauge fields which correspond to these isometries and they are also the only transformations the action is non-trivially invariant under. The result is therefore largely a negative one: the action is only gauge invariant if it is constructed from the standard Poincaré gauge fields and this invariance is lost when one tries to include BMS fields.

In the following section 5 we performed a κ-deformation of the BMS symmetry. Such deformed symmetries are expected to describe the structure of spacetimes at the Planck scale, where quantum gravity effects become important. After reviewing some necessary mathematical notions we showed how this deformation can be done in a constructive way using a twist deformation. Such a deformation leaves the algebra sector unchanged but leads to a deformed coproduct, which in turn implies a modified composition rule for Eigenvalues of many-particle states. We found that according to this modified rule the sum of two supertranslation or superrotation charges generically depends on charges from the Poincaré sector.

Finally, in section 6 we discussed black hole entropy, the information loss paradox and its relation to the BMS symmetry. In an attempt to better understand the microscopic origin of black hole entropy we first revisited the brick wall model by 't Hooft. By incorporating backreaction effects of Hawking radiation on the spacetime we were able to give a natural explanation for the appearance of the brick wall, which is a regulator that was introduced to obtain a finite result for the black hole entropy equal to the standard Bekenstein-Hawking expression. In our derivation the brick wall is identified with the quantum ergosphere, the region between the event and apparent horizon. In contrast to 't Hooft's original argument we obtain the standard expression for the black hole entropy with high accuracy without introducing an external parameter.

In the second part of this section we explained an argument recently proposed by Hawking et al. that is related to the BMS symmetry and potentially resolves the information loss paradox semi-classically. They argue that since there is an infinite number of conserved BMS charges, correlations are introduced during the black hole evaporation process between quanta of early and late radiation. These correlations could conceivably lead to a deviation from the thermal spectrum of the Hawking radiation and render the evaporation process unitary. Bousso et al. refute this argument by claiming that the modes which carry the BMS charges evolve trivially and that they completely decouple from modes carrying the Poincaré charges. Our results from the κ-deformation of the BMS symmetry showed that due to the deformed coproducts such a decoupling of Poincaré and BMS modes is not possible. Such effects become large only at the Planck scale and therefore go beyond the semi-classical treatment.

8 Appendix

8.1 Four-dimensional deformed antipode

The deformed antipode can be obtained from its defining property (5.6) and using the expressions for the coproduct (5.54)-(5.57) one obtains

$$
\begin{aligned}
S_{n,\varepsilon}(k_m) = &-k_m - \frac{i}{(1-2n)\sqrt{2}\kappa}\big((m+2n-1)k_0 S_{1-n+m,1-n} + m k_m S_{1-n,1-n}\big) \\
&+ \frac{i\varepsilon}{2(1-2n)\sqrt{2}\kappa}\Big(k_{1-2n}[(m-2n+1)S_{n+m,1-n} + (m+2n-1)S_{1-n+m,n}] \\
&+ \bar{k}_{1-2n}[(m-2n+1)A_{n+m,1-n} + (m+2n-1)A_{n,1-n+m}] \\
&- 2(1-2n-m)[k_{1-2n+m}S_{n,1-n} + \bar{k}_{1-2n+m}A_{n,1-n}]\Big) + O(\kappa^{-2}) \qquad (8.1)
\end{aligned}
$$

$$
\begin{aligned}
S_{n,\varepsilon}(\bar{k}_m) = &-\bar{k}_m - \frac{i}{(1-2n)\sqrt{2}\kappa}\big((m+2n-1)k_0 A_{1-n+m,1-n} + m k_m A_{1-n,1-n}\big) \\
&+ \frac{i\varepsilon}{2(1-2n)\sqrt{2}\kappa}\Big(k_{1-2n}[(m-2n+1)A_{n+m,1-n} + (m+2n-1)A_{1-n+m,n}] \\
&+ \bar{k}_{1-2n}[(m-2n+1)S_{n+m,1-n} + (m+2n-1)S_{n,1-n+m}] \\
&- 2(1-2n-m)[\bar{k}_{1-2n+m}S_{n,1-n} - k_{1-2n+m}A_{n,1-n}]\Big) + O(\kappa^{-2}) \qquad (8.2)
\end{aligned}
$$

$$
\begin{aligned}
S_{n,\varepsilon}(S_{pq}) = &-S_{pq} - \frac{i}{(1-2n)\sqrt{2}\kappa}(p+q)S_{pq}S_{1-n,1-n} \\
&+ \frac{i\varepsilon}{(1-2n)\sqrt{2}\kappa}\Big([(1-n-p)S_{p+1-2n,q} + (1-n-q)S_{q+1-2n,p}]S_{n,1-n} \\
&+ [(1-n-p)A_{p+1-2n,q} + (1-n-q)A_{q+1-2n,p}]A_{n,1-n}\Big) + O(\kappa^{-2}) \qquad (8.3)
\end{aligned}
$$

$$
\begin{aligned}
S_{n,\varepsilon}(A_{pq}) = &-A_{pq} - \frac{i}{(1-2n)\sqrt{2}\kappa}(p+q)A_{pq}S_{1-n,1-n} \\
&+ \frac{i\varepsilon}{(1-2n)\sqrt{2}\kappa}\Big([(1-n-p)A_{p+1-2n,q} + (1-n-q)A_{p,q+1-2n}]S_{n,1-n} \\
&+ [(1-n-p)S_{p+1-2n,q} + (1-n-q)S_{q+1-2n,p}]A_{n,1-n}\Big) + O(\kappa^{-2}). \qquad (8.4)
\end{aligned}
$$

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